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Geometric Aspects of the Abelian Modular Functions of Genus Four.

By Arthur B. Coble.*

Introduction.

A quadratic Cremona transformation with three F-points (fundamental points) at three nodes of a given rational plane sextic transforms the sextic into another rational sextic which of course is projectively distinct from the given one. By successive application of this process new sextics are obtained any one of which is the transform of the original one by a single Cremona transformation—the product of the successive quadratic transformations. The higher Cremona transformations which appear in this way are infinite in number comprising indeed all such transformations with ten or fewer F-points. The writer has shown (l. c. **4**) that this infinite number of transformations applied to a given sextic will produce only a finite number of projectively distinct transforms. Thus under Cremona transformation a given sextic is equivalent to $2^{13}.31.51$ projectively distinct types; and these types are permuted by such transformation according to a finite group isomorphic with a certain theta modular group of genus five. An entirely similar situation appears in the case of the ten nodes of the Cayley symmetroid—the quartic surface defined by the vanishing of a symmetric four-row determinant whose elements are linear forms. The quadratic transformation is replaced by a cubic transformation in space with four double F-points at the nodes of the symmetroid; and again under the infinite number of regular Cremona transformations thus generated the symmetroid is transformed into only a finite number of projectively distinct types. The types in this case are permuted under a group isomorphic to the group, reduced mod. 2, of integer linear transformations of the periods of the theta functions of genus four. The existence of these theta modular factor groups indicates a close relation of the ten-nodal configurations to the corresponding modular functions. A positive result in this direction is due to F. Schottky.† He has shown that one may define by certain combinations of abelian theta modular functions of

* These articles give the results of an investigation undertaken by the author in 1920 for the Carnegie Institution of Washington, D. C. For brief abstracts now appearing in the *Proceedings of the Natl. Acad. of Sciences*, see Vol. 7 (1921), p. 245 and p. 334; Vol. 9 (1923), p. 183.

† *Acta Math.*, Vol. 27 (1903), p. 235.

genus four the coördinates of a set of ten points in space which have a characteristic property of the nodes of the symmetroid. There remains however the goemetric problem of attaching to a given symmetroid curves of genus four which will define the modular functions.

The extensive geometric theory connected with these configurations has already been attacked from several directions. Cayley * has discussed the symmetroid Σ as a birational transform of the jacobian J of a web of quadrics (cf. **10**). Meyer † has discussed the relation of J to the rational plane sextic and mentions the occurrence of conter sextics (cf. **4**). Conner ‡ considers the mapping of J upon Σ and the connection of Σ with two paired rational space sextics which map from two space cubic curves C_1, C_2 defined by J (cf. **11**). It is my purpose in Part I of this memoir to give a comprehensive account of this theory. The simplest point of departure appears to be the figure of two space cubic curves (cf. **3**). The incidence condition of plane of the one and point of the other is a birationally general algebraic relation of genus four, $F=0$, which forms the basis of the algebraic discussion. Algebraic forms with digredient variables are constantly employed. The rational sextics themselves appear in tetrads of paired and counter sextics and the theory of their pespective rational curves is very useful.

In a general way those matters which depend upon a separation of the nodes of the sextic and symmetroid are reserved for Part II. Here the discontinuous groups associated with the nodal configurations are important. The work of Wirtinger,§ who develops the curve of genus four as the locus of vertices of diagonal triangles of coresidual four-points on a quartic curve of genus three, will be carried further. One would expect the modular functions of genus five associated with the sextic, and those of genus four associated with the symmetroid, to be related in some such way.

1. The Curve of Genus 4.

The curve of genus 4 has a unique canonical series g_3^6 and, when mapped upon an S_3 by a linear system of spreads which cut out this series, it becomes the normal curve of genus 4, a space sextic which is the complete intersection of a quadric and a cubic surface. In general the unique quadric on such a curve is a proper quadric whose points may be named by the binary parameters

* *Proc. London Math. Soc.*, Vol. 3 (1871), p. 19.

† *Apolarität und Rationale Curven*, pp. 320-47.

‡ This *Journal*, Vol. 37 (1915), p. 29.

§ *Math. Ann.*, Vol. 27 (1892), p. 261; *Untersuchungen über Thetafunctionen*, Leipzig (1895).

t, τ of the cross generators through them. Then the equation of the space sextic is given by the vanishing of the double binary form

$$(1) \qquad F \equiv (a\tau)^3 (at)^3$$

of order three in the digredient binary variables τ, t. Conversely such a form interpreted on a quadric gives rise to a space sextic cut out of the quadric by a cubic surface and therefore of genus 4. The form F with 16 coefficients depends upon 15 constants of which 6 may be removed by digredient transformation of t and τ. Thus F has 9 absolute constants which may be regarded as the algebraic moduli attached to the curve of genus 4. Rational or irrational invariants of the form F under digredient transformations we shall define to be algebraic modular functions for the curve. We shall make frequent use of algebraic forms in variables drawn from different domains and shall denote by the symbol

$$\begin{pmatrix} i_1, & i_2, & \cdots \\ k_1, & k_2, & \cdots \end{pmatrix}$$

an algebraic form of order i_1 in the variables of a space S_{k_1} of dimension k_1, of order i_2 in the variables of an S_{k_2}, etc. If $k_r = k_s$ the variables of S_{k_r} and S_{k_s} are to be regarded as digredient unless expressly restricted. Thus we refer to F as a form $\binom{33}{11}$.

The canonical series $g_3^{\,6}$ contains two special series $g_1^{\,3}$ coresidual with respect to each other in the canonical series. These are cut out by the pencils of planes on the generators of the quadric. We shall refer to them as the τ-triads or t-triads respectively. Thus when τ is fixed in $F = 0$ the three values of t thus determined will with the given τ locate a τ-triad cut out on the space curve by the τ-generator. There are 12 τ-triads with a double point or 12 τ-generators which touch the space sextic. Their parameters τ are the branch points of the algebraic function $t(\tau)$ determined by $F = 0$ or the roots of the discriminant of F regarded as a cubic in t. The binary 12-ic in τ which furnishes these branch points has likewise $12 - 3 = 9$ absolute constants. This raises at once the question as to the number (presumably finite) of forms F for given binary 12-ic in τ and as to the reciprocal relation of the 12-ics in τ and in t for the two algebraic functions $t(\tau)$ and $\tau(t)$ defined by $F = 0$.

In the particular case (subject to one condition) where the space sextic is cut out on a quadric cone by a cubic surface the algebraic discussion with the form F as a basis fails. Geometrically also this case differs so radically from the general case as to require separate treatment.

An especially symmetric case of the plane curve of genus 4 is the sextic with 6 nodes. Its cubic adjoints map it upon the space sextic but at the same time they map the plane into a cubic surface which contains the space sextic and directions at the nodes into a line-six (half of a double six) on the surface, whose lines are bisecants of the curve. The transition to a different line-six on the same cubic surface corresponds in the plane to a Cremona transform of the given plane sextic into a similar plane sextic.* Thus a plane sextic of genus 4 with 13 absolute projective constants, as one of its set of 72 projectively distinct Cremona transforms, determines a space sextic of genus 4 with 9 absolute constants and one of the ∞^4 cubic surfaces on the curve.

2. Comitants of Degree 3 of the $\binom{3\ 3}{1\ 1}$ Form F.

A complete system of comitants of the form F would certainly be hard to get and we shall for the moment be content with such a system to the limited degree 3. In the following sections we shall have immediate geometric interpretations for some of the comitants obtained here. We denote by $C^{(k)}_{i,j}$ a comitant of F of order i in τ, order j in t, and of degree k in the coefficients. In writing the explicit symbolic form of a comitant we give only the determinant factors. Reductions are accomplished by the use of the usual identity, $(ab)=(ac)+(cb)$ and the interchange of equivalent symbols. For degrees one and two we have at once:

$$\text{Degree one:}\quad F=C^{(1)}_{3,3}$$

$$\begin{aligned}\text{Degree two:}\quad & F^2=[C^{(1)}_{3,3}]^2,\ (ab)(\alpha\beta)\equiv C^{(2)}_{4,4},\ (ab)(\alpha\beta)^3\equiv C^{(2)}_{4,0}\\ & (ab)^3(\alpha\beta)\equiv C^{(2)}_{0,4},\ (ab)^3(\alpha\beta)^3\equiv C^{(2)}_{0,0},\ (\alpha\beta)^2\equiv C^{(2)}_{6,2},\\ & (ab)^2\equiv C^{(2)}_{2,6},\ (ab)^2(\alpha\beta)^2\equiv C^{(2)}_{2,2};\\ & (ab)=(\alpha\beta)=(ab)^2(\alpha\beta)=(ab)(\alpha\beta)^2\equiv 0,\\ & (ab)^3=(\alpha\beta)^3=(ab)^3(\alpha\beta)^2=(ab)^2(\alpha\beta)^3\equiv 0.\end{aligned}$$

As a check on completeness we observe that the forms retained have a total of 136 coefficients [thus F^2, a $C^{(2)}_{6,6}$ has 49, $C^{(2)}_{4,4}$ has 25, etc.] which is the number of linearly independent quadratic combinations of the 16 coefficients of F.

For the comitants of degree 3 we must have a totality of $16.17.18/1.2.3=816$ coefficients since there are no syzygies of this degree. The minimum weight of such a coefficient is 0, 0 for $(a_0{}^3\alpha_0{}^3)^3$, the maximum weight is 9, 9 for

* A. B. Coble, "Point Sets and Allied Cremona Groups," *Trans. Amer. Math. Soc.*, I, Vol. 16 (1915), p. 155; II, Vol. 17 (1916), p. 345; III, Vol. 18 (1917), p. 331; in particular III, § 1, p. 332. These papers are cited hereafter as "Cremona Groups."

$(a_1{}^3\alpha_1{}^3)^3$ in F^3. We easily find the number $n_{i,j}$ of cubic combinations of the coefficients of weight (i, j) to be as follows:

$$n_{0,0}, \cdots, n_{0,4} = 1,\ 1,\ 2,\ 3,\ 3; \quad n_{1,1}, \cdots n_{1,4} = 2,\ 4,\ 6,\ 7;$$
$$n_{2,2}, \cdots, n_{2,4} = 8,\ 12,\ 14; \ n_{3,3},\ n_{3,4} = 19,\ 22; \quad n_{4,4} = 26.$$

The remaining numbers $n_{i,j}$ are found from the relations

$$n_{i,j} = n_{j,i}; \quad n_{9-i,j} = n_{i,j}.$$

We denote by $p_{i,k}$ the product $F.C^{(2)}_{i-3,k-3}$ and construct the following table in which the first column contains a value of $n_{i,j}$, the second column the new comitants which must exist in order that their leading coefficients together with proper coefficients of earlier comitants shall make up the number $n_{i,j}$ of terms of the required weight, and the third column contains the limits for the first and second weights of the new comitants.

$n_{0,0} = 1$	F^3	$0 \cdot\cdot 9, 0 \cdot\cdot 9$	$n_{2,4} = 14$	$C^{(3)}_{5,1}$	$2 \cdot\cdot 7, 45$
$n_{0,2} = 2$	$p_{9,5}$	$0 \cdot\cdot 9, 2 \cdot\cdot 7$	$n_{3,0} = 3$	$C^{(3)}_{3,9}$	$3 \cdot\cdot 6, 0 \cdot\cdot 9$
$n_{0,3} = 3$	$C^{(3)}_{9,3}$	$0 \cdot\cdot 9, 3 \cdot\cdot 6$	$n_{3,1} = 6$	$p_{3,7}$	$3 \cdot\cdot 6, 1 \cdot\cdot 8$
$n_{1,1} = 2$	$p_{7,7}$	$1 \cdot\cdot 8, 1 \cdot\cdot 8$	$n_{3,2} = 12$	$C^{(3)}_{3,5}, C'^{(3)}_{3,5}$	$3 \cdot\cdot 6, 2 \cdot\cdot 7$
$n_{1,2} = 4$	$C^{(3)}_{7,5}$	$1 \cdot\cdot 8, 2 \cdot\cdot 7$	$n_{3,3} = 19$	$p_{3,3}, C^{(3)}_{3,3}, C'^{(3)}_{3,3}$	$3 \cdot\cdot 6, 3 \cdot\cdot 6$
$n_{1,3} = 6$	$p_{7,3}$	$1 \cdot\cdot 8, 3 \cdot\cdot 6$	$n_{3,4} = 22$	$C^{(3)}_{3,1}$	$3 \cdot\cdot 6, 45$
$n_{1,4} = 7$	$C^{(3)}_{7,1}$	$1 \cdot\cdot 8, 4 \cdot\cdot 5$	$n_{4,1} = 7$	$C^{(3)}_{1,7}$	$45, 1 \cdot\cdot 8$
$n_{2,0} = 2$	$p_{5,9}$	$2 \cdot\cdot 7, 0 \cdot\cdot 9$	$n_{4,2} = 14$	$C^{(3)}_{1,5}$	$45, 2 \cdot\cdot 7$
$n_{2,1} = 4$	$C^{(3)}_{5,7}$	$2 \cdot\cdot 7, 1 \cdot\cdot 8$	$n_{4,3} = 22$	$C^{(3)}_{1,3}$	$45, 3 \cdot\cdot 6$
$n_{2,2} = 8$	$p_{5,5}, C^{(3)}_{5,5}$	$2 \cdot\cdot 7, 2 \cdot\cdot 7$	$n_{4,4} = 26$	$C^{(3)}_{1,1}$	$45, 45$
$n_{2,3} = 12$	$C^{(3)}_{5,3}, C'^{(3)}_{5,3}$	$2 \cdot\cdot 7, 3 \cdot\cdot 6$			

We locate thus 18 irreducible comitants of degree 3. There remains the problem of expressing all the symbolic products in terms of these 18 and of products of earlier ones. This is accomplished by using the symbolic identities. The results are tabulated as follows:

The complete system of F for the degree 3 is

$$\begin{aligned}
C^{(1)}_{3,3} &= F = (a\tau)^3(\alpha t)^3 \\
C^{(2)}_{4,4} &= (ab)(\alpha\beta)(a\tau)^2(b\tau)^2(\alpha t)^2(\beta t)^2 \\
C^{(2)}_{2,6} &= (ab)^2(a\tau)(b\tau)(\alpha t)^3(\beta t)^3 \\
C^{(2)}_{6,2} &= (\alpha\beta)^2(a\tau)^3(b\tau)^3(\alpha t)(\beta t) \\
C^{(2)}_{0,4} &= (ab)^3(\alpha\beta)(\alpha t)^2(\beta t)^2 \\
C^{(2)}_{4,0} &= (ab)(\alpha\beta)^3(a\tau)^2(b\tau)^2 \\
C^{(2)}_{2,2} &= (ab)^2(\alpha\beta)^2(a\tau)(b\tau)(\alpha t)(\beta t)
\end{aligned}$$

$$
\begin{aligned}
C^{(2)}_{0,0} &= (ab)^3(\alpha\beta)^3 \\
C^{(3)}_{3,9} &= (ab)^2(bc)(a\tau)(c\tau)^2(\alpha t)^3(\beta t)^3(\gamma t)^3 \\
C^{(3)}_{9,3} &= (\alpha\beta)^2(\beta\gamma)(a\tau)^3(b\tau)^3(c\tau)^3(\alpha t)(\gamma t)^2 \\
C^{(3)}_{5,7} &= (ac)^2(\alpha\beta)(a\tau)(b\tau)^3(c\tau)(\alpha t)^2(\beta t)^2(\gamma t)^3 \\
C^{(3)}_{7,5} &= (ab)(\alpha\gamma)^2(a\tau)^2(b\tau)^2(c\tau)^3(\alpha t)(\beta t)^3(\gamma t) \\
C^{(3)}_{1,7} &= (ab)(bc)^2(ac)(\alpha\beta)(a\tau)(\alpha t)^2(\beta t)^2(\gamma t)^3 \\
C^{(3)}_{7,1} &= (ab)(\alpha\beta)(\beta\gamma)^2(\alpha\gamma)(a\tau)^2(b\tau)^2(c\tau)^3(\alpha t) \\
C^{(3)}_{5,5} &= (ab)^2(\alpha\gamma)^2(a\tau)(b\tau)(c\tau)^3(\alpha t)(\beta t)^3(\gamma t) \\
C^{(3)}_{3,5} &= (ab)^2(ac)(\alpha\beta)(\alpha\gamma)(b\tau)(c\tau)^2(\alpha t)(\beta t)^2(\gamma t)^2 \\
C^{(3)}_{5,3} &= (ab)(ac)(\alpha\beta)^2(\alpha\gamma)(a\tau)(b\tau)^2(c\tau)^2(\beta t)(\gamma t)^2 \\
C'^{(3)}_{3,5} &= (bc)^2(ab)(\alpha\beta)(\alpha\gamma)(a\tau)^2(c\tau)(\alpha t)(\beta t)^2(\gamma t)^2 \\
C'^{(3)}_{5,3} &= (ab)(ac)(\beta\gamma)^2(\alpha\beta)(a\tau)(b\tau)^2(c\tau)^2(\alpha t)^2(\gamma t) \\
C^{(3)}_{1,5} &= (ab)(ac)^2(bc)(\alpha\beta)^2(b\tau)(\alpha t)(\beta t)(\gamma t)^3 \\
C^{(3)}_{5,1} &= (ab)^2(\alpha\beta)(\alpha\gamma)^2(\beta\gamma)(a\tau)(b\tau)\ (c\tau)^3(\beta t) \\
C^{(3)}_{3,3} &= (ab)(ac)(bc)(\alpha\beta)(\alpha\gamma)(\beta\gamma)(a\tau)(b\tau)(c\tau)(\alpha t)(\beta t)(\gamma t) \\
C_{3,3}^{(3)} &= (ab)^2(bc)(\alpha\beta)^2(\beta\gamma)(a\tau)(c\tau)^2(\alpha t)(\gamma t)^2 \\
C^{(3)}_{1,3} &= (ab)^2(ac)(bc)(\alpha\beta)^2(\beta\gamma)(c\tau)(\alpha t)(\gamma t)^2 \\
C^{(3)}_{3,1} &= (ab)^2(bc)(\alpha\beta)^2(\alpha\gamma)(\beta\gamma)(a\tau)(c\tau)^2(\gamma t) \\
C^{(3)}_{1,1} &= (ab)^2(ac)(bc)(\alpha\beta)(\alpha\gamma)^2(\beta\gamma)(c\tau)(\beta t)
\end{aligned}
$$

In terms of this complete system the expressions for the remaining symbolic products of degree three are

Order $9, 9$: F^3.

$9, 7$: $(\alpha\beta) \equiv 0$.

$9, 5$: $(\alpha\beta)^2 = 2(\alpha\beta)(\alpha\gamma) = p_{9,5}$.

$9, 3$: $(\alpha\beta)^3 = (\alpha\beta)(\alpha\gamma)(\beta\gamma) \equiv 0$.

$9, 1$: $(\alpha\beta)^2(\alpha\gamma)(\beta\gamma) \equiv 0$.

$7, 7$: $(ab)(\alpha\beta) = 2(ab)(\alpha\gamma) = p_{7,7}$.

$7, 5$: $(ab)(\alpha\beta)^2 = (ab)(\alpha\gamma)(\beta\gamma) \equiv 0$; $(ab)(\alpha\beta)(\alpha\gamma) = C^{(3)}_{7,5}$.

$7,3$: $(ab)(\alpha\beta)^3 = 2(ab)(\alpha\gamma)^3 = -2(ab)(\alpha\beta)^2(\beta\gamma) =$
$2(ab)(\alpha\gamma)^2(\alpha\beta) = p_{7,3}$;
$(ab)(\alpha\gamma)^2(\beta\gamma) = (ab)(\alpha\beta)(\alpha\gamma)(\beta\gamma) \equiv 0$.

$7, 1$: $(ab)(\alpha\beta)^2(\alpha\gamma)(\beta\gamma) \equiv 0$.

$5, 5$: $(ab)(\alpha\beta)^2 = 2(ab)^2(\alpha\beta)(\alpha\gamma) = 2(ab)(ac)(\alpha\beta)^2 = p_{5,5}$;
$2(ab)^2(\alpha\gamma)(\beta\gamma) = 2(ab)(ac)(\beta\gamma)^2 = -p_{5,5} + 2\, C^{(3)}_{5,5}$;
$4(ab)(ac)(\alpha\beta)(\alpha\gamma) = 3p_{5,5} - 2\, C^{(3)}_{5,5}$;
$4(ab)(ac)(\alpha\beta)(\beta\gamma) = p_{5,5} - 2\, C^{(3)}_{5,5}$.

$5, 3$: $(ab)^2(\alpha\beta)^3 = (ab)^2(\alpha\beta)(\alpha\gamma)(\beta\gamma) = (ab)(ac)(\beta\gamma)^3 =$
$(ab)(ac)(\alpha\beta)(\alpha\gamma)(\beta\gamma) \equiv 0$;

$$(ab)^2(a\gamma)^3 = (ab)(ac)(a\beta)^3 = C^{(3)}_{5,3} + C'^{(3)}_{5,3};$$
$$(ab)^2(a\beta)^2(\beta\gamma) = (ab)^2(a\gamma)^2(a\beta) = C^{(3)}_{5,3} - C'^{(3)}_{5,3};$$
$$(ab)^2(a\gamma)^2(\beta\gamma) = -2(ab)(ac)(a\gamma)^2(\beta\gamma) = 2\, C'^{(3)}_{5,3}.$$

5, 1: $(ab)^2(a\beta)^2(a\gamma)(\beta\gamma) = 2(ab)(ac)(a\beta)^2(a\gamma)(\beta\gamma) = -(ab)(ac)(a\beta)(a\gamma)(\beta\gamma)^2 = 2\, C^{(3)}_{5,1}.$

3, 3 $(ab)^3(a\beta)^3 = -2(ab)^3(a\beta)^2(\beta\gamma) = -2(ab)^2(bc)(a\beta)^3 = p_{3,3};$
$-2(ab)^3(a\gamma)^2(\beta\gamma) = -(ab)^3(a\beta)(a\gamma)(\beta\gamma) =$
$-2(ab)^2(bc)(\gamma a)^3 = 2(ab)^2(bc)(a\beta)(a\gamma)(\beta\gamma) =$
$-(ab)(ac)(bc)(a\beta)^3 = 2(ab)(ac)(bc)(a\beta)^2(\beta\gamma) = C^{(3)}_{3,3}:$
$2(ab)^3(\beta\gamma)^2(a\beta) = 2(ab)^2(bc)(\beta\gamma)^3 = p_{3,3} - 2\, C^{(3)}_{3,3}:$
$2(ab)^3(a\gamma)^3 = p_{3,3} - 3\, C^{(3)}_{3,3};\ 2(ab)^2(bc)(a\beta)^2(a\gamma) = -p_{3,3} + 2\, C'^{(3)}_{3,3};$
$2(ab)^2(bc)(\beta\gamma)^2(a\beta) = -2\, C'^{(3)}_{3,3} + C^{(3)}_{3,3};$
$2(ab)^2(bc)(a\gamma)^2(\beta\gamma) = p_{3,3} - 2\, C'^{(3)}_{3,3};$
$2(ab)^2(bc)(\beta\gamma)^2(a\gamma) = 2(ab)^2(bc)(a\gamma)^2(\beta a) = p_{3,3} - 2\, C'^{(3)}_{3,3} - C^{(3)}_{3,3}.$

3, 1: $(ab)^3(a\beta)^2(a\gamma)(\beta\gamma) = (ab)(ac)(bc)(a\beta)^2(a\gamma)(\beta\gamma) = 0;$
$-2(ab)^3(a\beta)(a\gamma)^2(\beta\gamma) = 3(ab)^2(bc)(a\beta)(a\gamma)^2(\beta\gamma) = 6(ab)^2(bc)(a\beta)(a\gamma)(\beta\gamma)^2 = 6\, C^{(3)}_{3,1}.$

1, 1: $(ab)^2(ac)(bc)(a\beta)^2(a\gamma)(\beta\gamma) = 2\, C^{(3)}_{1,1}.$

On interchanging Greek and italic letters in the relations just given, all symbolic products of degree 3 are expressed in terms of the complete system.

3. The Figure of Two Cubic Curves in Space. Reciprocity between the Forms F, $\bar{F}$.

H. S. White has introduced * for other purposes the interpretation of the equation $F = 0$ as the incidence condition of the point τ of the space cubic curve $C_1(\tau)$ and the plane t of the space cubic $C_2(t)$. Thus when we take, by proper choice of the coördinate system in space, the cubic curve $C_1(\tau)$ as

(1) $x_0 = \tau^3,\quad x_1 = 3\tau^2,\quad x_2 = 3\tau,\quad x^3 = 1,$ or in planes as
$\xi_0 = 1,\quad \xi_1 = -\tau,\quad \xi_2 = \tau^2,\quad \xi_3 = -\tau^3;$

then any other cubic curve $C_2(t)$ in planes has the form

$$\begin{aligned}
\xi_0 &= r_{00}t^3 + 3r_{01}t^2 + 3r_{02}t + r_{03}, \\
\xi_1 &= r_{10}t^3 + 3r_{11}t^2 + 3r_{12}t + r_{13}, \\
\xi_2 &= r_{20}t^3 + 3r_{21}t^2 + 3r_{22}t + r_{23}, \\
\xi_3 &= r_{30}t^3 + 3r_{31}t^2 + 3r_{32}t + r_{33}.
\end{aligned} \tag{2}$$

* *Proceedings of the National Academy of Sciences*, Vol. 2 (1916), p. 337.

The incidence condition of point $x(\tau)$ of $C_1(\tau)$ and plane $\xi(t)$ of $C_2(t)$ is

(3) $$F=(a\tau)^3(\alpha t)^3=0 \quad (a_0^{3-i}\,a_1^{\,i}\,\alpha_0^{3-j}\,\alpha_1^{\,j}=r_{ij}).$$

Obviously for given $C_1(\tau)$ the totality of cubic curves exhausts the totality of forms F. Each cubic curve has 12 projective constants and the pair has $24-15=9$ absolute constants which is the number of absolute constants in F.

There is however a dual incidence condition of plane $\xi(\tau)$ of $C_1(\tau)$ and point $x(t)$ of $C_2(t)$. The parametric equation of $C_2(t)$ in points is

(4) $$\begin{aligned} Rx_0 &= R_{00}-R_{01}t+R_{02}t^2-R_{03}t^3,\\ Rx_1 &= R_{10}-R_{11}t+R_{12}t^2-R_{13}t^3,\\ Rx_2 &= R_{20}-R_{21}t+R_{22}t^2-R_{23}t^3,\\ Rx_3 &= R_{30}-R_{31}t+R_{32}t^2-R_{33}t^3, \end{aligned}$$

where R_{ij} is the cofactor of r_{ij} in the determinant

(5) $$R=\left| r_{ij} \right|.$$

The incidence condition of plane $\xi(\tau)$ of $C_1(\tau)$ and point $x(t)$ of $C_2(t)$ is

(6) $$\bar{F}\equiv(\bar{a}\tau)^3(\bar{\alpha}t)^3=0,$$

where the expanded form of $\bar{F}$ is

(7) $$\bar{F}=\begin{vmatrix} r_{00} & r_{01} & r_{02} & r_{03} & 1\\ r_{10} & r_{11} & r_{12} & r_{13} & -\tau\\ r_{20} & r_{21} & r_{22} & r_{23} & \tau^2\\ r_{30} & r_{31} & r_{32} & r_{33} & -\tau^3\\ -1 & t & -t^2 & t^3 & 0 \end{vmatrix}=\sum_{i=0}^{3}\sum_{j=0}^{3}(-1)^{i+j}R_{ij}\tau^i t^j.$$

Thus for the symbolic coefficients $\bar{a}\bar{\alpha}$ we have the values

(8) $$\bar{a}_0^{3-i}\,\bar{a}_1^{\,i}\,\bar{\alpha}_0^{3-j}\,\bar{\alpha}_1^{\,j}=\bar{r}_{ij}=-(1)^{i+j}R_{3-i,3-j}/\tbinom{3}{i}.\tbinom{3}{j}.$$

The symbolic form of $\bar{F}$ in terms of F is found to be

(9) $$6\bar{F}=6(\bar{a}\tau)^3(\bar{\alpha}t)^3=C_{3,3}^{(3)}=$$
$$(ab)(ac)(bc)(\alpha\beta)(\alpha\gamma)(\beta\gamma)(a\tau)(b\tau)(c\tau)(\alpha t)(\beta t)(\gamma t).$$

The reciprocity between the forms F, $\bar{F}$ is brought out by the fact that the covariant $\bar{F}$, formed for $\bar{F}$ as a ground form is again F; more specifically

(10) $$\bar{F}(\bar{F})=R^2.F/3^4.$$

The invariant R has the values,

$$(11) \quad 6(a\bar{a})^3(\alpha\bar{\alpha})^3 = (ab)(ac)(bc)(ad)(bd)(cd)(\alpha\beta)(\alpha\gamma)(\beta\gamma)(\alpha\delta)(\beta\delta)(\gamma\delta) = 24R.$$

The reciprocity mentioned fails when $R = 0$. Then the coefficients of τ in F, 4 cubics in t, are linearly dependent. When expressed in terms of 3 cubics, the coefficients of the three in F are three cubics in τ and the equation $F = 0$ can be interpreted as the incidence condition of point τ on a rational *plane* cubic C_1 with line t of a rational *plane* cubic C_2. Two such rational plane cubics have 8 rather than 9 absolute constants. In this case the form $\bar{F}$ factors into two cubics, $(\bar{a}\tau)^3.(\bar{\alpha}t)^3$, the binary cubics apolar respectively to all the line sections of C_1 and point sections of C_2. Now all the (usually non-vanishing) odd-odd transvectants of the double form $\bar{F}$ with itself will vanish identically whence in the general case they contain the factor R and are further of the second degree in the coefficients of F. From their degree and orders they can be identified at once in the list of comitants in Q namely

$$(12) \quad \begin{aligned} {}^*\tfrac{2}{9}(b\tau)^4(\beta t)^4 &\equiv C^{(2)}_{4,4} = (aa')(\alpha\alpha')(a\tau)^2(a'\tau)^2(\alpha t)^2(\alpha' t)^2, \\ \tfrac{4}{3}(c\tau)^4 &\equiv C^{(2)}_{4,0} = (aa')(\alpha\alpha')^3(a\tau)^2(a'\tau)^2, \\ \tfrac{4}{3}(\gamma t)^4 &\equiv C^{(2)}_{0,4} = (aa')^3(\alpha\alpha')(\alpha t)^2(\alpha' t)^2, \\ 8\delta &\equiv C^{(2)}_{0,0} = (aa')^3(\alpha\alpha')^3. \end{aligned}$$

The interpretation of these forms with respect to the space cubic curves $C_1(\tau)$ and $C_2(t)$ is as follows. The points τ_1, τ_2 of C_1 are respectively on two triads of planes of C_2 whose parameters are given by the cubics $(a\tau_1)^3(\alpha t)^3 = 0$, $(a'\tau_1)^3(\alpha' t)^3 = 0$. A point on the line joining τ_1, τ_2 is on a triad of planes of C_2 determined by a member of the pencil $(a\tau_1)^3(\alpha t)^3 + \lambda(a'\tau_2)^3(\alpha' t)^3 = 0$. If this member has a double root t the point of the line τ_1, τ_2 is on a tangent t of C_2 whence the line meets 4 tangents of C_2 whose parameters are the Jacobian of the pencil i. e. $(a\tau_1)^3(a'\tau_2)^3(\alpha\alpha')(\alpha t)^2(\alpha' t)^2 = 0$. By interchanging equivalent symbols and factoring out $(\tau_1\tau_2)/2$ this reduces to

$$(aa')(\alpha\alpha')(\alpha t)^2(\alpha' t)^2[(a\tau_1)^2(a'\tau_2)^2 + (a\tau_1)(a\tau_2)(a'\tau_1)(a'\tau_2) + (a\tau_2)^2(a'\tau_1)^2].$$

If now $\tau_1 = \tau_2 = \tau$ the line τ_1, τ_2 becomes the tangent τ of C_1 and the form becomes $3\,C^{(2)}_{4,4}$ whence

(13) *The condition that tangent τ of C_1 and tangent t of C_2 be incident is that* $(b\tau)^4(\beta t)^4 = 0$.

---

* **We shall henceforth use primes and seconds to denote equivalent symbols so as to have extra letters for the comitants as needed.**

If the pencil of cubics above is apolar to itself the line τ_1, τ_2 is in the null-system of the curve C_2. Then

$$(a\tau_1)^3(a'\tau_2{}^3)(\alpha\alpha')^3 =$$
$$\tfrac{1}{2}(\tau_1\tau_2).(aa')(\alpha\alpha')^3[(a\tau_1)^2(a'\tau_2)^2 + \cdots + (a\tau_2)^2(a'\tau_1)^2] = 0.$$

Again letting $\tau_1 = \tau_2$ we find that

(14) *The equation, $(c\tau)^4 = 0$, determines the four tangents τ of C_1 which are in the null-system of C_2; $(\gamma t)^4 = 0$, the four tangents t of C_2 which are in the null-system of C_1.*

Since the equation of the null-system of C_2 is of degree two in its coefficients r_{ij} and that of C_1 is numerical the apolarity condition of the two must be an invariant of degree two whence

(15) *The equation, $\delta = 0$, expresses that the null systems of the curves C_1 and C_2 are apolar.*

These interpretations given as they are in terms of self dual concepts such as the tangent and null system of the curves C_1, C_2 should evidently be self dual and therefore the comitants involved should have the same rôle for the dual forms F, $\bar{F}$.

The reciprocity betwen the dual forms F, $\bar{F}$ is due algebraically to the fact that their coefficients are respectively the one- and three-row minors of R. The four self dual comitants of the second degree above should therefore involve the two-row minors of R. Indeed we find that the 36 coefficients of the four comitants, $(b\tau)^4(\beta t)^4$, $(c\tau)^4$, $(\gamma t)^4$, δ are linearly independent in the 36 two-row minors of R. Their explicit forms are as follows where

$$(16) \qquad \binom{jl}{ik} \equiv \begin{vmatrix} r_{ij} & r_{il} \\ r_{kj} & r_{kl} \end{vmatrix} :$$

$$\begin{aligned}
\tfrac{1}{9}(b\tau)^4(\beta t)^4 = {} & \tau^4\{\tbinom{01}{01}t^4 + 2\tbinom{02}{01}t^3 + [\tbinom{03}{01} + 3\tbinom{12}{01}]t^2 + 2\tbinom{13}{01}t + \tbinom{23}{01}] \\
& + 2\tau^3\{\tbinom{01}{02}t^4 + 2\tbinom{02}{02}t^3 + [\tbinom{03}{02} + 3\tbinom{12}{02}]t^2 + 2\tbinom{13}{02}t + \tbinom{23}{02}\} \\
& + \tau^2\{[\tbinom{01}{03} + 3\tbinom{01}{12}]t^4 + 2[\tbinom{02}{03} + 3\tbinom{02}{12}]t^3 + [\tbinom{03}{03} + 3\tbinom{03}{12} + \\
& \qquad 3\tbinom{12}{03} + 9\tbinom{12}{12}]t^2 + 2[\tbinom{13}{03} + 3\tbinom{13}{12}]t + [\tbinom{23}{03} + 3\tbinom{23}{12}]\} \\
& + 2\tau\{\tbinom{01}{13}t^4 + 2\tbinom{02}{13}t^3 + [\tbinom{03}{13} + 3\tbinom{12}{13})]t^2 + 2\tbinom{13}{13}t + \tbinom{23}{13}\} \\
& + \{\tbinom{01}{23}t^4 + 2\tbinom{02}{23}t^3 + [\tbinom{03}{23} + 3\tbinom{12}{23}]t^2 + 2\tbinom{13}{23}t + \tbinom{23}{23}\}. \\
\tfrac{2}{3}(\gamma t)^4 = {} & [\tbinom{01}{03} - 3\tbinom{01}{12}]t^4 + 2[\tbinom{02}{03} - 3\tbinom{02}{12}]t^3 + [\tbinom{03}{03} - 3\tbinom{03}{12} \\
& \qquad + 3\tbinom{12}{03} - 9\tbinom{12}{12}]t^2 + 2[\tbinom{13}{03} - 3\tbinom{13}{12}]t + [\tbinom{23}{03} - 3\tbinom{23}{12}]. \\
\tfrac{2}{3}(c\tau)^4 = {} & [\tbinom{03}{01} - 3\tbinom{12}{01}]\tau^4 + 2[\tbinom{03}{02} - 3\tbinom{12}{02}]\tau^3 + [\tbinom{03}{03} + 3\tbinom{03}{12} \\
& \qquad - 3\tbinom{12}{03} - 9\tbinom{12}{12}]\tau^2 + 2[\tbinom{03}{13} - 3\tbinom{12}{13}]\tau + [\tbinom{03}{23} - 3\tbinom{12}{23}]. \\
4\delta = {} & \tbinom{03}{03} - 3\tbinom{03}{12} - 3\tbinom{12}{03} + 9\tbinom{12}{12}.
\end{aligned}$$

Since the 36 two-row minors are functions of only 16 elements there must be numerous relations among them which will be indicated by the existence of syzygies among the four comitants (16). We shall now determine such quadratic relations and the corresponding quadratic syzygies. We note in the first place that there are no linear relations among the 36 minors. For such a relation would be of the form $\Sigma C_{ijkl}\binom{jl}{ik}=0$ where the coefficients C_{ijkl} are numerical. If we multiply the columns in order by $\lambda_0, \cdots, \lambda_3$ and the rows in order by $\mu_0, \cdots, \mu_3$ we should have a new determinant for which this relation, now $\Sigma\, \lambda_j\lambda_l\mu_i\mu_k C_{ijkl}\binom{jl}{ik}=0$, would hold for all values of λ, μ. Since this is an identity in λ, μ, $C_{ijkl}\binom{jl}{ik}=0$ or $C_{ijkl}=0$. An extension of this argument shows that in any identity among the 36 minors which is not a sum of simpler identities every term consisting of a product of minors must involve a particular row or a particular column the same number of times.

In a quadratic relation among the minors each term can involve not more than four rows each once, or three rows with one used twice, or two rows used twice, since if one row were used three or four times a minor in the term would necessarily vanish. Also if less than four rows are used then necessarily four columns must be used since there is no relation among the 9 two-row minors of a 3-row determinant. If then we take the columns 0, 1, 2, 3 and the rows 0, 1, 0, 1 each term must be of the form $\binom{jl}{01}\binom{j'l'}{01}$ where $j, l, j', l' = 0, 1, 2, 3$ and each term must occur as in the Laplace expansion of a determinant with two pairs of equal rows. Hence for the six choices of ik we have six identities

$$\text{(a)} \qquad \binom{01}{ik}\binom{23}{ik}+\binom{02}{ik}\binom{31}{ik}+\binom{03}{ik}\binom{12}{ik}=0$$

and a similar set of six for four distinct rows and two pairs of like columns. In the next case for four choices of the equal row h and 3 choices of another pair ik we have 12 identities of the form

$$\text{(b)} \qquad \binom{01}{hi}\binom{23}{hk}+\binom{01}{hk}\binom{23}{hi}+\binom{02}{hi}\binom{31}{hk}+\binom{02}{hk}\binom{31}{hi}$$
$$+\binom{03}{hi}\binom{12}{hk}+\binom{03}{hk}\binom{12}{hi}=0,$$

which arises from the Laplace expansion of a determinant with two equal rows. Any identity containing these terms must be of this form since for $i=k$ it must reduce to (a). There is a similar set of 12 identities for two like columns. Finally if all four rows are used with all four columns we have the six Laplace expansions of R namely

$$\text{(c)} \qquad R = \Sigma(\overline{gh})(\overline{ik}) = \Sigma(\overline{gi})(\overline{kh}) = \Sigma(\overline{gk})(\overline{hi}) = \\ = \Sigma(\underline{gh})(\underline{ik}) = \Sigma(\underline{gi})(\underline{kh}) = \Sigma(\underline{gk})(\underline{hi}).$$

In this way we obtain 5 identities. Hence

(17) *There are* 41 *quadratic relations among the* 36 *two-row minors of a four-row determinant.*

We seek now those quadratic covariants of the four comitants (16) which vanish due to the existence of these 41 quadratic relations among their coefficients. First of all we shall list these quadratic covariants. From the powers and products of the four comitants we get one comitant of each of the orders (8, 8), (8, 0), (0, 8), (0, 0), (8, 4), (4, 8), (4, 0), (0, 4) and two comitants of the orders (4, 4). To these we add the non-vanishing transvectants of each comitant with itself and with the others. These comprise one comitant of each of the orders (8, 4), (4, 8), (8, 0), (0, 8), (6, 6), (6, 4), (4, 6), (6, 2), (2, 6), (4, 2), (2, 4), (2, 2); and three comitants of each of the orders (4, 4), (4, 0), (0, 4), (0, 0).

The total number of terms in these quadratic covariants is 666 which is just the number of quadratic combinations of the 36 minors. There must be certain linear relations among these quadratic covariants such that the coefficients of their terms will furnish the 41 quadratic relations.

In order to find them we pick out such a covariant as will have for part or all of one of its coefficients a term which appears in a quadratic identity and by adding properly chosen other covariants we make this coefficient coincide with a quadratic identity. If one such coefficient is made up of quadratic identities this must be true of all. For example the product $\binom{01}{01}\binom{23}{01}$ occurs in one identity. On referring to (16) we see that these minors occur as the coefficients of $\tau^4 t^4$ and τ^4 in $(b\tau)^4(\beta t)^4$ and therefore the product will be part of the coefficient of τ^8 in $(b\tau)^4(b'\tau)^4(\beta\beta')^4$. In fact

$$(b\tau)^4(b'\tau)^4(\beta\beta')^4 = 162\left\{\tbinom{01}{01}\tbinom{23}{01} + \tbinom{02}{01}\tbinom{31}{01} + \tfrac{1}{12}\left[\tbinom{03}{01} + 3\tbinom{12}{01}\right]^2\right\}\tau^8 + \ldots$$

$$[(c\tau)^4]^2 = \tfrac{9}{4}\left\{\left[\tbinom{03}{01} - 3\tbinom{12}{01}\right]^2\right\}\tau^8 + \ldots;$$

whence in

$$(b\tau)^4(b'\tau)^4(\beta\beta')^4 - 6[(c\tau)^4]^2$$

the coefficient of τ^8 is a determinant identity and this covariant vanishes identically.

The coefficient of τ^8 furnishes the identity with rows 01, 01; that of τ^7 the identity with rows 01, 02; while that of τ^6 contains the identities with rows 01, 03 as well as that with rows 02, 02. To separate these identities we

observe that their terms occur in the coefficient of τ^4 of $(bb')^2(b\tau)^2(b'\tau)^2(\beta\beta')^4$ and proceeding as before we find that

$$(\beta\beta')^4(bb')^2(b\tau)^2(b'\tau)^2 - 6(cc')^2(c\tau)^2(c'\tau)^2 \equiv 0.$$

Similarly we find the syzygies

$$(b\tau)^4(\beta\gamma)^4 - 6\delta.(c\tau)^4 \equiv 0,$$
$$(bb')^4(\beta\beta')^4 \equiv 6(cc')^4 \equiv 6(\gamma\gamma')^4 \equiv 36\delta.^2$$

Hence we have shown that

(18) *The four comitants* (16) *whose coefficients are formed from the two-row minors of R are connected by the following syzygies of the second degree*

$$(b\tau)^4(b'\tau)^4(\beta\beta')^4 \equiv 6[(c\tau)^4]^2, \qquad (bb')^4(\beta t)^4(\beta' t)^4 \equiv 6[(\gamma t)^4]^2,$$
$$(bb')^2(b\tau)^2(b'\tau)^2(\beta\beta')^4 \equiv 6(cc')^2(c\tau)^2(c'\tau)^2,$$
$$(bb')^4(\beta\beta')^2(\beta t)^2(\beta' t)^2 \equiv 6(\gamma\gamma')^2(\gamma t)^2(\gamma' t)^2,$$
$$(b\tau)^4(\beta\gamma)^4 \equiv 6\delta.(c\tau)^4, \qquad (bc)^4(\beta t)^4 \equiv 6\delta.(\gamma t)^4,$$
$$(bb')^4(\beta\beta.)^4 = 6(cc')^4 = 6(\gamma\gamma')^4 = 36\delta^2.$$

The 41 *coefficients of these syzygies furnish the* 41 *linearly independent relations of the second degree among the two-row minors of R. These syzygies determine for given $(b\tau)^4(\beta t)^4$ the three other comitants to within sign and are unaltered by the change of sign of any two of these three comitants.*

It is of value to know those comitants of F of degree k which when formed for $\bar{F}$ have a degree less than $3k$ due to the fact that a power of R separates out. Thus we know that the comitants (16) of degree two for F are also of degree two when formed for $\bar{F}$ and that the comitant $C_{3,3}^{3}$ of **2** when formed for $\bar{F}$ is effectively of degree one since R^2 factors out. For the remaining comitants of degree three we observe that all of those which can be found by transvection of one of the four comitants (16) with F itself will have coefficients which are linear in the two-row minors of R. Such comitants therefore when formed for $\bar{F}$ will have a factor R and hence will be effectively of degree five only in the coefficients of F.

We form therefore a table of these transvectants which are labelled as follows:

$$T_{ij} = [(a\tau)^3(\alpha t)^3,\ (a'a'')(\alpha'\alpha'')(a'\tau)^2(a''\tau)^2(\alpha' t)^2(\alpha'' t)^2]^{i,j},$$
(19) $$T_i = [(a\tau)^3(\alpha t)^3,\ (a'a'')(\alpha'\alpha'')^3(a'\tau)^2(a''\tau)^2]^i.$$

and as a result of the calculation we find that

$$
(20) \qquad \begin{array}{lll}
T_{01} = -C^{(3)}_{7,5}, & 2T_{11} = p_{5,5} - C^{(3)}_{5,5}, & 36T_{22} = 24C^{(3)}_{3,3} + p_{3,3}, \\
6T_{02} = p_{7,3}, & 6T_{12} = -2C^{(3)}_{5,3} - C'^{(3)}_{5,3}, & 2T_{23} = C^{(3)}_{3,1}, \\
T_{03} = -C^{(3)}_{7,1}, & 2T_{13} = -3C^{(3)}_{5,1}, & 2T_{33} = 3C^{(3)}_{1,1}, \\
T_1 = C^{(3)}_{5,1} + C'^{(3)}_{5,1}, & 6T_2 = p_{3,3} - 6C^{(3)}_{3,3}, & T_3 = 3C^{(3)}_{1,3},
\end{array}
$$

(21) *All the irreducible comitants of F of degree three except $C^{(3)}_{9,3}$, $C^{(3)}_{3,9}$, $C'^{(3)}_{3,3}$ are linear in the two-row minors of R and when formed for $\bar{F}$ are of effective degree five in the coefficients of F except for $C^{(3)}_{3,3}$ whose effective degree is one.*

The four comitants (16) are used again in **8** for the study of certain combinants determined by F. Meanwhile we return to the cubic curves $C_1(\tau)$, $C_2(t)$ for the interpretation of other comitants of F of the second degree, interpretations which however are no longer self dual.

4. A Set of Four Mutually Related Rational Plane Sextics and their Covariant Conics $K(\tau)$, $K(t)$.

Let $C_1(\tau)$ and $C_2(t)$ be the two cubic space curves regarded as point loci; $\bar{C}_1(\tau)$, $\bar{C}_2(t)$ the same curves regarded as loci of planes. Let Q_1, Q_2 be the two nets of point quadrics on C_1, C_2 respectively; $\bar{Q}_1$, $\bar{Q}_2$ the two nets of quadric envelopes on $\bar{C}_1$, $\bar{C}_2$ respectively. The pencils of the net Q_i are the pencils on C_i and a bisecant of C_i; the pencils of the net $\bar{Q}_i$ are the pencils on $\bar{C}_i$ and an axis of $\bar{C}_i$.

The net of quadrics Q_i will cut the curve C_j $(i, j = 1, 2;\ i \neq j)$ in an involution of hexads of points, an I_2^6. An I_2^6 on a binary domain may be visualized as the line sections of a projectively definite rational plane sextic. Thus we find a tetrad of rational sextics, namely: $S_1(\tau)$ whose line sections are cut out on $C_1(\tau)$ by the net Q_2; $S_2(t)$ cut out on $C_2(t)$ by the net Q_1; $\bar{S}_1(\tau)$ cut out on $\bar{C}_1(\tau)$ by the net $\bar{Q}_2$; and $\bar{S}_2(t)$ cut out on $\bar{C}_2(t)$ by the net $\bar{Q}_1$. The parameters of a node t_1, t_2 of $S_2(t)$ are a neutral pair of the I_2^6 whence the points t_1, t_2 of $C_2(t)$ lie on a pencil of the net Q_1 and therefore lie on a bisecant of C_1 i. e. a common bisecant of C_1, C_2. Conversely a common bisecant determines such a neutral pair of the I_2^6 whence

(1) *The curves C_1, C_2 have ten common bisecants which determine on C_1, C_2 the nodal parameters of $S_1(\tau)$, $S_2(t)$ respectively; dually the curves $\bar{C}_1$, $\bar{C}_2$ have ten common axes which determine on $\bar{C}_1$, $\bar{C}_2$ the nodal parameters of $\bar{S}_1(\tau)$, $\bar{S}_2(t)$ respectively.*

We wish now to show that any given rational plane sextic determines the pair of cubic curves C_1, C_2 and thereby the other three members of the tetrad.

We observe first that the nets Q_1 and Q_2 have no common quadrics. For there are but ∞^5 cubic curves on a given quadric and ∞^7 on a given net Q_1. Since for given net Q_1 on C_1, C_2 is any one of ∞^{12} cubic curves it will in general lie on no quadric of Q_1. Hence the linear system $Q_1 + Q_2$ formed from the nets Q_1, Q_2 is of dimension 5 or there is a web of ∞^3 quadric envelopes $\bar{Q}$ apolar to both nets Q_1, Q_2. Now, given $S_2(t)$, one of its line sections determines on $C_2(t)$ a hexad of points cut out by a quadric defined to within members of the net Q_2. Thus the ∞^2 line sections of $S_2(t)$ will determine the system $Q_1 + Q_2$ and therefore the apolar web $\bar{Q}$. But Reye * has shown that in the system apolar to a web $\bar{Q}$ there are precisely two nets Q_1, Q_2 which are on cubic curves C_1, C_2. Thus C_1 is uniquely determined and thereby the tetrad of rational sextics. Hence

(2) *The rational sextics of the plane can be arranged in tetrads* $\begin{matrix} S_1(\tau), S_2(t) \\ \bar{S}_1(\tau), \bar{S}_2(t) \end{matrix}$ *in such a way that given any one the other three are projectively determined.*

If the tetrad is arrayed as in (2) we shall say that two sextics in a row of the array are *paired sextics;* two in a column are *counter sextics;* and any other two are *diagonal sextics.*

The line sections of $S_2(t)$ in the plane π are in one-to-one and projective correspondence with the quadrics of the net Q_1. There is in Q_1 a quadratic system consisting of the cones on $C_1(\tau)$ which are in one-to-one correspondence with their vertices, the points τ of C_1. Hence there is on π a conic $K(\tau)$ whose lines correspond to these cones. To a pencil of lines in π on a point p there corresponds a pencil of quadrics in Q_1 which contains two cones which correspond to the two tangents of $K(\tau)$ on p. If in particular p is a node of $S_2(t)$ the pencil in Q_1 is the pencil on a common bisecant of C_1, C_2 and the two cones have nodes at τ_1, τ_2, the points of intersection with C_1 of this bisecant, and the parameters of a node of the paired sextic $S_1(\tau)$. Hence

(3) *Given a rational plane sextic, $S_2(t)$, there exists in its plane a covariant conic, $K(\tau)$, such that the ten pairs of parameters on $K(\tau)$ of the ten nodes of $S_2(t)$ furnish the nodal parameters of the sextic $S_1(\tau)$ paired with $S_2(t)$.*

The converse is perhaps more striking.

(4) *If with reference to a norm conic $K(\tau)$ we mark the ten points determined by the ten pairs of nodal parameters of a rational sextic $S_1(\tau)$ then these ten points are the nodes of another rational sextic $S_2(t)$ which is the sextic paired with $S_1(\tau)$.*

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* *Jour. für Math.*, Vol. 82 (1877), pp. 78-9.

We remark here that, since a rational sextic has only 9 absolute constants, only six pairs of nodal parameters can be chosen at random whereas in the plane eight nodes may be chosen at random and there remains one degree of freedom for the choice of the ninth node.* The relation between the conditions on the position of the nodes in the plane and the conditions on the nodal parameters on the curve, which for a single sextic would be much involved, becomes according to (3), (4) remarkably simple for the paired sextics.

The sextic $S_2(t)$ and its covariant conic $K(\tau)$ are so related that a tangent τ of $K(\tau)$ cuts out 6 points t of $S_2(t)$ and a point t of $S_2(t)$ is on two tangents τ of $K(\tau)$. This $(6, 2)$ relation in t, τ can be obtained from the cubic curves $C_1(\tau)$, $C_2(t)$ as follows. The condition that plane τ of C_1 is on point t of C_2 is $(\bar{a}\tau)^3(\bar{a}t)^3 = 0$. The pencil of Q_1 on t of C_2 contains the two nodal quadrics with vertices at τ_1, τ_2 the meets of C_1 and the bisecant from t to C_1. Then τ_1, τ_2 are the Hessian pair of the three planes of C_1 on t i. e. the Hessian of the cubic $(\bar{a}\tau)^3(\bar{a}t)^3$ in τ. Hence

(5) *The parametric equations of the tetrad* (2) *of rational sextics in Darboux coördinates referred to their covariant conics* $K(\tau)$, $K(t)$ *respectively as norm conics are*

$$(aa')^2(at)(a't)(a\tau)^3(a'\tau)^3 = 0 \qquad (\bar{a}\bar{a}')^2(\bar{a}\tau)(\bar{a}'\tau)(\bar{a}t)^3(\bar{a}'t)^3 = 0,$$
$$(\bar{a}\bar{a}')^2(\bar{a}t)(\bar{a}'t)(\bar{a}\tau)^3(\bar{a}'\tau)^3 = 0, \qquad (aa')^2(a\tau)(a'\tau)(at)^3(a't)^3 = 0.$$

Another property of the paired sextics appears from their connection with the space cubics. We have taken in **3**(1) the curve $C_1(\tau)$ in normal form. Referred to it a binary cubic $(c\tau)^3$ determines either a point in space through which there pass the three planes of $C_1(\tau)$ whose parameters are determined by $(c\tau)^3 = 0$; or a plane in space on which there lie the three points of $C_1(\tau)$ with parameters $(c\tau)^3 = 0$. Similarly with respect to $C_2(t)$ a binary cubic determines either a point or a plane of space. The transition from the one reference curve to the other is effected by the formulae (1), (2), and (4) of **3**. Thus the point $(c\tau)^3$ with reference to $C_1(\tau)$ is according to **3**(1) the point $-c_0\xi_3 + 3c_1\xi_2 - 3c_2\xi_1 + c_3\xi_0$ and this from **3**(2) becomes with reference to $C_2(t)$ the point $-c_0(r_{30}t^3 + 3r_{31}t^2 + \cdots) + 3c_1(r_{20}t^3 + 3r_{21}t^2 + \cdots) \cdots = (ac)^3(at)$.[3] Similarly the plane $(c\tau)^3$ with reference to $C_1(\tau)$ is $c_0x_0 + c_1x_1 + c_2x_2 + c_3x_3 = 0$ and this according to **3**(4) becomes with reference to $C_1(\tau)$ the plane $\{c_0(R_{00} - R_{01}t \cdots) + c_1(R_{10} - R_{11}t \cdots) \cdots\}/R = (c\bar{a})^3(\bar{a}t)^3/R$. By using the solved forms of **3**(2), (4) we pass from the

* Cf. A. B. Coble, "The Ten Nodes of the Rational Sextic, etc.," *Amer. Jour. of Math.*, Vol. 41 (1919), p. 251.

point $(\gamma t)^3$ or plane $(\gamma t)^3$ with reference to $C_2(t)$ to the same point or plane with reference to $C_1(\tau)$. Thus

(6) *The forms F, $\bar{F}$ represent the direct and inverse linear transformations of the parametric coördinate system in space determined by $C_1(\tau)$ into that determined by $C_2(t)$. Specifically we find that*

referred to $C_1(\tau)$		referred to $C_2(t)$
the point $(c\tau)^3 = 0$	*is*	*the point* $(ac)^3(at)^3 = 0$;
the plane $(c\tau)^3 = 0$	"	*the plane* $(c\bar{a})^3(\bar{a}t)^3/R = 0$;
the point $(\bar{a}\tau)^3(\gamma\bar{a})^3/R = 0$	"	*the point* $(\gamma t)^3 = 0$;
the plane $(a\tau)^3(a\gamma)^3 = 0$	"	*the plane* $(\gamma t)^3 = 0$.

Let t_1, t_2, t_3 be three points of the sextic, $S_2(t)$, on a line. Then there is a quadric q_1 of the net Q_1 on the three points t_1, t_2, t_3 of $C_2(t)$ which with the net Q_2 on $C_2(t)$ determines a web which meets the plane of t_1, t_2, t_3 in the net of conics on these three points. Hence one quadric $q_1 + q_2$ of the web contains the plane of t_1, t_2, t_3. This plane meets $C_1(\tau)$ in three points τ_1, τ_2, τ_3 which also are points in which the quadric q_2 of the net Q_2 meets $C_1(\tau)$ whence τ_1, τ_2, τ_3 are three points of the paired sextic $S_1(\tau)$ on a line. Evidently the relation between the linear triad t on $S_2(t)$ and the linear triad τ on $S_1(\tau)$ is mutual whence

(7) *There is a one-to-one correspondence between the linear triads on two paired sextics; if $(\gamma t)^3 = 0$ is the linear triad on $S_2(t)$ $\{\bar{S}_2(t)\}$ then $(a\tau)^3(a\gamma)^3 = 0\{\bar{a}\tau)^3(\gamma\bar{a})^3 = 0\}$ is the linear triad on the paired sextic $S_1(\tau)\{\bar{S}_1(\tau)\}$; if $(c\tau)^3 = 0$ is the linear triad on $S_1(\tau)\{\bar{S}_1(\tau)\}$ then $(c\bar{a})^3(\bar{a}t)^3 = 0\{(ac)^3(at)^3 = 0\}$ is the linear triad on the paired sextic $S_2(t)\{\bar{S}_2(t)\}$.*

5. The Perspective Cubics of the Plane Rational Sextic. The Form $\binom{1\,1\,3}{2\,1\,1}$.

In this section we consider the perspective cubics of the rational sextic $\bar{S}_2(t)$ of the tetrad $\mathbf{4}(2)$ first as they occur in connection with the pair of cubic space curves and secondly as they are defined by the general form $\binom{1\,1\,3}{2\,1\,1}$.

(1) $$(\pi x)(a\tau)(at)^3 = 0.$$

where x is a ternary variable, and τ, t digredient binary variables as before. Such a form has $3.2.4$ coefficients and therefore $23 - 8 - 3 - 3 = 9$ absolute constants.

The ∞^1 planes τ of C_1 cut out on a particular plane τ_0 of C_1 a line conic

which we identify with $K(\tau)$. Any point x in the plane τ_0 is determined by the two tangents of $K(\tau)$ and these arise from the two planes τ_1, τ_2 of C_1 on these tangents. The two planes meet in an axis l_x $(x = \tau_1, \tau_2)$ which cuts the plane τ_0 in the point x. Thus the congruence of axes l_x of C_1 determines a projective correspondence among all the planes of C_1.

Into this figure we inject the curve $C_2(t)$. The planes t of C_2 cut each plane τ of C_1 in a rational line cubic

$$(2) \qquad \begin{array}{lll} & (\pi x)(a\tau)(at)^3 = (a\tau_1)(a\tau_2)(a\tau)(at)^3 & \text{when} \\ & x_0 = \tau_1\tau_2, \quad x_1 = \tau_1 + \tau_2, \quad x_2 = 1; & \text{and} \\ K(\tau): & x_0 = \tau^2, \quad x_1 = 2\tau, \quad x_2 = 1; \quad 4x_0x_2 - x_1^2 = 0. & \end{array}$$

For according to **4**(6) the plane t of $C_2(t)$ is the plane $(a\tau)^3(at)^3$ referred to $C_1(\tau)$ and the point τ_1, τ_2, τ is on this plane if $(a\tau_1)(a\tau_2)(a\tau)(at)^3 = 0$.

Selecting a fixed plane τ_0 of C_1 as a base the ∞^1 (for variable τ) rational cubic envelopes (2) on the ∞^1 planes τ are projected upon τ_0 by means of the axes l_x to form a family (with parameter τ) of ∞^1 rational line cubics in τ_0 (with parameter t). Since for fixed t there is one axis l_x in the plane t of C_2 which for variable τ is met by all the lines (2) it follows that the cubic curves (2) in τ_0 are all *perspective* (i. e. line t of the envelope is on point t of the rational curve) to the rational curve cut out on τ_0 by the ∞^1 axes of C_1 which lie in planes t of C_2. Since for given t these axes are determined by the quadratics $(aa')^2(a\tau)(a'\tau)(at)^3(a't)^3 = 0$ this rational curve is the sextic $\bar{S}_2(t)$ referred to its covariant conic $K(\tau)$. In particular the ten common axes of C_1, C_2 which are on two planes t cut the plane τ_0 in the nodes of $\bar{S}_2(t)$. Also as x runs over a line in τ_0 the axes l_x are generators of a quadric in $\bar{Q}_1$ which has 6 planes in common with C_2 and the 6 axes l_x in these planes mark on $\bar{S}_2(t)$ the points where the line meets $\bar{S}_2(t)$. Hence

(3) *Upon any plane τ_0 of C_1 the axes of C_1 which lie in the planes of $C_2(t)$ cut out the rational sextic $\bar{S}_2(t)$, the planes of C_1 cut out the covariant conic $K(\tau)$ of $\bar{S}_2(t)$, and the planes of C_2 cut the ∞^1 planes of C_1 in ∞^1 rational cubic envelopes which are projected upon τ_0 by the congruence of axes of C_1 into the ∞^1 perspective cubics of $\bar{S}_2(t)$. The nodes of $\bar{S}_2(t)$ are cut out on τ_0 by the ten common axes of C_1, C_2.*

The equation of the form (2) in terms of the coefficients r_{ij} of F is

$$(3) \qquad \begin{aligned} & t^3[(r_{00}x_0 + r_{10}x_1 + r_{20}x_2)\tau + (r_{10}x_0 + r_{20}x_1 + r_{30}x_2)] \\ + & 3t^2[(r_{01}x_0 + r_{11}x_1 + r_{21}x_2)\tau + (r_{11}x_0 + r_{21}x_1 + r_{31}x_2)] \\ + & 3t[(r_{02}x_0 + r_{12}x_1 + r_{22}x_2)\tau + (r_{12}x_0 + r_{22}x_1 + r_{32}x_2)] \\ + & [(r_{03}x_0 + r_{13}x_1 + r_{23}x_2)\tau + (r_{13}x_0 + r_{23}x_1 + r_{33}x_3)] = 0 \end{aligned}$$

for which the norm conic $K(\tau)$ is isolated.

We shall however now take the form more generally as

$$(4)\qquad (\pi x)(a\tau)(\alpha t)^3 \equiv (\tau a_0 + b_0)t^3 + 3(\tau a_1 + b_1)t^2 + 3(\tau a_2 + b_2)t + (\tau a_3 + b_3)$$

where a_i, b_i $(i = 0, 1, 2, 3)$ are general linear forms in the ternary variable x. For fixed τ and variable t we have in (4) the parametric equation of a rational cubic envelope; for fixed t and variable τ we have in (4) the parametric equation of a point whose ternary equation is

$$1^\circ \qquad (\pi\pi'\xi)(aa')(\alpha t)^3(\alpha' t)^3 = 0.$$

For variable t this point runs over the rational sextic $\bar{S}_2(t)$ to which the cubic envelopes are perspective.

The systematic study of perspective curves was begun by W. Stahl.* Further developments are given in an article by the writer † based on the incidence condition of point of the one curve with line of the other. It there appears (l. c., p. 351(124)) that conversely the general sextic has ∞^1 perspective line cubics whose parametric equations contain a parameter linearly so that the form (4) is projectively defined by $\bar{S}_2(t)$.

Since sextic 1° and cubic (4) are perspective the incidence condition of point t of 1° and line t_1 of (4) must contain the factor $(t - t_1)$. This condition is

$$\begin{aligned}(\pi\pi'\pi'')(aa')(a''\tau)(\alpha t)^3(\alpha' t)^3(\alpha'' t_1)^3 = \\ \tfrac{1}{3}(\pi\pi'\pi'')\{(\alpha t)^3(\alpha' t)^3(\alpha'' t_1)^3(aa')(a''\tau) + \\ +(\alpha' t)^3(\alpha'' t)^3(\alpha t_1)^3(a'a'')(a\tau) + (\alpha'' t)^3(\alpha t)^3(\alpha' t_1)^3(a''a)(a'\tau)\} = 0.\end{aligned}$$

From the identity

$$(aa')(a''\tau) + (a'a'')(a\tau) + (a''a)(a'\tau) \equiv 0$$

eliminate the last term in the above condition after which by interchange of α, α''; a, a''; π, π'' it becomes

$$\begin{aligned}&\tfrac{2}{3}(\pi\pi'\pi'')(\alpha t)^3\{(\alpha' t)^3(\alpha'' t_1)^3 - (\alpha' t_1)^3(\alpha'' t)^3\}(aa')(a''\tau) = \\ (5)\quad &\tfrac{2}{3}(tt_1)\,.\,(\pi\pi'\pi'')(aa')(\alpha'\alpha'')(a''\tau)(\alpha t)^3\{(\alpha' t)^2(\alpha'' t_1)^2 \\ &+ (\alpha' t)(\alpha' t_1)(\alpha'' t)(\alpha'' t_1) + (\alpha' t_1)^2(\alpha'' t)^2\} = 0.\end{aligned}$$

Hence

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* *Math. Ann.*, Vol. 38 (1891).

† A. B. Coble, "Symmetric Binary Forms and Involutions (III)," *Amer. Jour. of Math.*, Vol. 32 (1920), p. 333; in particular § 15, p. 350. Other references are given there.

(6) *The incidence condition of point t of $\bar{S}_2(t)$ and line t_1 of the perspective cubic* (4) *is the condition* (5) *of degree three in the coefficients. When* (4) *is taken in the canonical form* (2) *this condition (to within the factor $\frac{2}{3}(tt_1)$) expressed as a Gordan expansion in terms of the comitants of degree* 3 *of* 2 *becomes*

$$-3[C^{(3)}_{1,7}]_{t_1^2}-{}^{27}\!/_7[C^{(3)}_{1,5}]_{t_1}\cdot(tt_1)+{}^{9}\!/_5[C^{(3)}_{1,3}]\cdot(tt_1)^2=0.$$

Thus the equation $C^{(3)}_{1,7}=0$ furnishes for given τ the 7 contacts of the perspective cubic with the rational sextic. According to **3**(21) the corresponding condition formed for $\bar{F}$ is R times a comitant of degree 5.

If we set

$$(7)\qquad C^{(3)}_{1,7}\equiv(l\tau)(\lambda t)^7,\quad C^{(3)}_{1,5}\equiv(m\tau)(\mu t)^5,\quad C^{(3)}_{1,3}\equiv(n\tau)(\nu t)^3$$

then, given t' there is a perspective cubic for which τ is determined by $(l\tau)(\lambda t')^7=0$ which touches $\bar{S}_2(t)$ at t'. If we set this value of τ in the incidence condition and let $t_1=t'$ then an additional factor (tt') must appear leaving a residual factor of degree 4 in t and 8 in t' which is the incidence condition of tangent t' and further point t of $\bar{S}_2(t)$. This condition is

$$(8)\qquad \begin{aligned}&-3(ll')(\lambda\lambda')(\lambda t')^2(\lambda't')^2\,[2(\lambda t)^4(\lambda't')^4+2(\lambda t)^3(\lambda t')(\lambda' t)(\lambda't')^3\\&\quad+(\lambda t)^2(\lambda' t)^2(\lambda t')^2(\lambda't')^2]\\&\qquad-{}^{27}\!/_7(ml)(\mu t)^4(\mu t')(\lambda t')^7+{}^{9}\!/_5(nl)(\nu t)^3(\lambda t')^7.\,(tt')=0.\end{aligned}$$

Setting $t'=t$ we have the flex equation of $\bar{S}_2(t)$

$$(9)\qquad -15(ll')(\lambda\lambda')(\lambda t)^6(\lambda' t)^6-{}^{27}\!/_7(ml)(\mu t)^5(\lambda t)^7=0.$$

The comitants of the second degree of the form (4) are, in addition to 1° the following:

2° $(\pi\pi'\xi)(aa')(\alpha\alpha')^2(\alpha t)^2(\alpha' t)^2$

3° $(\pi\pi'\xi)(\alpha\alpha')(a\tau)(a'\tau)(\alpha t)^2(\alpha' t)^2$

4° $(\pi\pi'\xi)(\alpha\alpha')^3(a\tau)(a'\tau)$

5° $(\pi x)(\pi' x)(aa')(\alpha\alpha')(\alpha t)^2(\alpha' t)^2$

6° $(\pi x)(\pi' x)(aa')(\alpha\alpha')^3$

7° $(\pi x)(\pi' x)(a\tau)(a'\tau)(\alpha\alpha')^2(\alpha t)(\alpha' t)$.

Their non-symbolic forms in terms of the 8 lines a_i, b_i of (4) are

1° $(a_0b_0\xi)t^6+3[(a_0b_1\xi)+(a_1b_0\xi)]+3[(a_0b_2\xi)+3(a_1b_1\xi)+(a_2b_0\xi)]t^4$
$\quad+[(a_0b_3\xi)+9(a_1b_2\xi)+9(a_2b_1\xi)+(a_3b_0\xi)]t^3$
$\quad+3[(a_1b_3\xi)+3(a_2b_2\xi)+(a_3b_1\xi)]t^2+3[(a_2b_3\xi)+(a_3b_2\xi)]t+(a_3b_3\xi)$;

2° $[(a_0b_2\xi)-2(a_1b_1\xi)+(a_2b_0\xi)]t^2+[(a_0b_3\xi)-(a_1b_2\xi)-(a_2b_1\xi)+(a_3b_0\xi)]t$
$\quad+[(a_1b_3\xi)-2(a_2b_2\xi)+(a_3b_1\xi)]$;

$$
\begin{aligned}
3^\circ\ \ & \{(a_0a_1\xi)t^4+2(a_0a_2\xi)t^3+[(a_0a_3\xi)+3(a_1a_2\xi)]t^2+2(a_1a_3\xi)\,t+(a_2a_3\xi)\} \\
& +\tau\{[(a_0b_1\xi)+(b_0a_1\xi)]t^4+2[(a_0b_2\xi)+(b_0a_2\xi)\,t^3]+[(a_0b_3\xi)+(b_0a_3\xi) \\
& +3(a_1b_2\xi)+3(b_1a_2\xi)]t^2+2[(a_1b_3\xi)+(b_1a_3\xi)]t+[(a_2b_3\xi)+(b_2a_3\xi)]\} \\
& +\tau^2\{(b_0b_1\xi)t^4+2(b_0b_2\xi)t^3+[(b_0b_3\xi)+3(b_1b_2\xi)]t^2+2(b_1b_3\xi)\,t+(b_2b_3\xi)\}\,;
\end{aligned}
$$

$$
\begin{aligned}
4^\circ\ \ & [(a_0a_3\xi)-3(a_1a_2\xi)]+\tau[(a_0b_3\xi)+(b_0a_3\xi)-3(b_1a_2\xi)-3(a_1b_2\xi)] \\
& \qquad\qquad\qquad\qquad\qquad\qquad +\tau^2[(b_0b_3\xi)-3(b_1b_2\xi)]\,;
\end{aligned}
$$

$$
\begin{aligned}
5^\circ\ \ & (a_0b_1-a_1b_0)t^4+2(a_0b_2-a_2b_0)t^3+(a_0b_3-a_3b_0+3a_1b_2-3a_2b_1)t^2 \\
& \qquad\qquad\qquad\qquad +2(a_1b_3-a_3b_1)t+(a_2b_3-a_3b_2)\,;
\end{aligned}
$$

$$
6^\circ\ \ a_0b_3-a_3b_0-3a_1b_2+3a_2b_1\,;
$$

$$
\begin{aligned}
7^\circ\ \ & 2\{[(a_0a_2-a_1^2)t^2+(a_0a_3-a_1a_2)t+(a_1a_3-a_2^2)]+\tau[(a_0b_2-2a_1b_1+a_2b_0)t^2 \\
& \quad +(a_0b_3+b_0a_3-a_1b_2-a_2b_1)t+(a_1b_3-2a_2b_2+a_3b_1)] \\
& \quad +\tau^2[(b_0b_2-b_1^2)t^2+(b_0b_3-b_1b_2)t+(b_1b_3-b_2^2)].
\end{aligned}
$$

All of these comitants except 2° have immediate geometric interpretations. Thus 3° is the point equation of the perspective line cubic. Also 4° is a point conic with parameter τ, the locus of points where the three cusp tangents of the perspective cubics meet. To interpret the last three forms we note that on a point x there is a pencil (parameter τ) of triads (t) of lines to the perspective cubics; and 5° is the jacobian of this pencil or the four double lines of the pencil whence it furnishes the parameters t of the point x on the four perspective cubics which pass through x; 6° is the self-apolarity invariant of the pencil; and 7° is the hessian of a particular τ-member of the pencil.

The most important comitant of degree 3 is

$$
\begin{aligned}
8^\circ\ \ & (\pi\pi'\pi'')(a\tau)(a'\tau)(a''\tau)(aa')(aa'')(a'a'')(at)(a't)(a''t)= \\
& -\tfrac{1}{3}(\pi\pi'\pi'')(a\tau)(a'\tau)(a''\tau)(aa')^3(a''t)^3= \\
& \quad \{(a_0a_1a_2)t^3+(a_0a_1a_3)t^2+(a_0a_2a_3)t+(a_1a_2a_3)\} \\
& +\tau\{[(b_0a_1a_2)+(a_0b_1a_2)+(a_0a_2b_3)]t^3+[(b_0a_1a_3)+(a_0b_1a_3)+(a_0a_1b_3)]t^2 \\
& \quad +[(b_0a_2a_3)+\cdots\qquad\qquad\qquad]t+[(b_1a_2a_3)+\cdots\qquad\qquad\qquad]\} \\
& +\tau^2\{[(a_0b_1b_2)+(b_0a_1b_2)+(b_0b_1a_2)]t^3+[(a_0b_1b_3)+(b_0a_1b_3)+(b_0b_1a_3)]t^2 \\
& \quad +[a_0b_2b_3)+\cdots\qquad\qquad\qquad]t+[(a_1b_2b_3)+\cdots\qquad\qquad\qquad]\} \\
& +\tau^3\{(b_0b_1b_2)t^3+(b_0b_1b_3)t^2+(b_0b_2b_3)t+(b_1b_2b_3)\}.
\end{aligned}
$$

For given τ this is the cubic apolar to all the point-sections of the corresponding cubic envelope and furnishes the parameters of its three cusps.

Since the cusps of the rational cubic envelope cut out on the plane τ of C_1 by planes t of C_2 are the points where C_2 cuts this plane it is clear that 8° is the incidence condition of plane τ and point t; and therefore when the perspective cubic is taken in the form (3) this incidence condition is $F=$

$(\bar{a}\tau)^3(\bar{a}t)^3=0$. Hence 8° is a general (3, 3) form and is birationally a general algebraic relation of genus 4. We shall now find the equation of this cusp locus.

If, for given τ, x is a cusp, then (4) is a perfect cube in t and its hessian as to t vanishes identically. Hence for x a cusp the form 7° vanishes identically in t and we obtain thereby three equations from which τ may be eliminated. This eliminant is a sextic in x whose equation derived from the non-symbolic form of 7° is the determinant

$$(10) \qquad \Delta_{a,b} \equiv \begin{vmatrix} \begin{vmatrix} a_0 & a_1 \\ a_1 & a_2 \end{vmatrix} & \begin{vmatrix} a_0 & b_1 \\ a_1 & b_2 \end{vmatrix} + \begin{vmatrix} b_0 & a_1 \\ b_1 & a_2 \end{vmatrix} & \begin{vmatrix} b_0 & b_1 \\ b_1 & b_2 \end{vmatrix} \\ \begin{vmatrix} a_0 & a_2 \\ a_1 & a_3 \end{vmatrix} & \begin{vmatrix} a_0 & b_2 \\ a_1 & b_3 \end{vmatrix} + \begin{vmatrix} b_0 & a_2 \\ b_1 & a_3 \end{vmatrix} & \begin{vmatrix} b_0 & b_2 \\ b_1 & b_3 \end{vmatrix} \\ \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} & \begin{vmatrix} a_1 & b_2 \\ a_2 & b_3 \end{vmatrix} + \begin{vmatrix} b_1 & a_2 \\ b_2 & a_3 \end{vmatrix} & \begin{vmatrix} b_1 & b_2 \\ b_2 & b_3 \end{vmatrix} \end{vmatrix} = 0.$$

Since $\Delta_{a,b}$ is of genus 4 it must have six nodes at which the distinct perspective cubics τ_1, τ_2 have cusps t_1, t_2. Then the three equations for the determination of τ are of rank one so that all the first minors of $\Delta_{a,b}$ vanish. The matrix formed by the first and last columns has two determinants with a common row which give rise to two quartic curves with 16 common points from which must be taken the four points determined by the conics of the common row. Thus the matrix defines 12 points. But the first column above vanishes at three points (the cusps of $\tau=0$) and the last column also vanishes for three points (the cusps of $\tau=\infty$) whence there remain 6 points which are the nodes x for which all first minors vanish. As to the constants in the curve we have four lines a which may be taken to be $\pm x_0 \pm x_1 \pm x_2$. There remain four lines b with 12 coefficients or, on allowing for the common factor of proportionality, 11 absolute constants. Thus the curve and the two triads of cusps, $\tau=0$, $\tau=\infty$, have 11 absolute constants or the curve itself has 9. Hence

(11) *The locus of cusps of perspective cubics of* $\bar{S}_2(t)$ *is a birationally general sextic of genus* 4 *whose equation is* $\Delta_{a,b}=0$. *The triads of cusps lie in one of the two* g_1^3*'s, the* τ*-triads, on the sextic. The six nodes of the sextic are the points for which the first minors of* $\Delta_{a,b}$ *vanish and these first minors furnish the* 9 *linearly independent adjoint quartic curves of the sextic.*

We shall find later in **9**(1) the distribution of the t-triads of the sextic $\Delta_{a,b}$.

We shall now determine the canonical adjoints of the sextic $\Delta_{a,b}$ of genus four. Through any point x there pass ∞^1 triads of tangents t of the ∞^1 perspective cubics τ. A particular triad of the pencil is given by (4) and the jacobian of the pencil by 5°. If x is a node of the sextic the pencil of triads has two perfect cubes, say t_1^3, t_2^3, and the jacobian is $t_1^2\, t_2^2$, a binary quartic apolar to every binary cubic of the pencil. This is the only type of pencil of cubics such that each cubic is apolar to the jacobian of the pencil. Hence if x is a node of $\Delta_{a,b}$ the cubic in t given by (4) for any τ is apolar to the quartic in t given by 5°. On forming this apolarity condition we get

$$9^\circ \qquad (\pi x)(\pi' x)(\pi'' x)(aa')(\alpha a')(a\alpha'')^2(a'\alpha'')(a't)(\alpha''\tau) = 9\begin{vmatrix} a_0 & a_1 & a_2 & \tau \\ a_1 & a_2 & a_3 & -\tau t \\ b_0 & b_1 & b_2 & -1 \\ b_1 & b_2 & b_3 & t \end{vmatrix}.$$

(12) *The web of adjoint cubic curves of the sextic $\Delta_{a,b}$ is furnished by the form 9°. For fixed τ and variable t we have the pencil of adjoint cubics on the cusp triad of the perspective cubic τ.*

Another way to write the form 9° is

$$\begin{aligned} &X_0 t + X_1 + X_2\tau t + X_3\tau \equiv \\ &t(a_3\overline{10} + a_2\overline{02} + a_1\overline{21}) + \tau t(b_3\overline{10} + b_2\overline{02} + b_1\overline{21}) \\ (13)\qquad &+(a_0\overline{23} + a_1\overline{31} + a_2\overline{12}) + \tau(b_0\overline{23} + b_1\overline{31} + b_2\overline{12}), \\ &\text{where } \overline{ij} = (a_i b_j - a_j b_i). \end{aligned}$$

Then the equation of the sextic $\Delta_{a,b}$ is

$$\begin{aligned} (14)\qquad \Delta_{a,b} &= X_0 X_3 - X_1 X_2 \\ &= \overline{21}^3 + 2\,\overline{01}\,\overline{12}\,\overline{23} + \overline{31}\,\overline{12}\,\overline{20} + \overline{10}\,\overline{03}\,\overline{32} + \overline{10}\,\overline{13}^2 + \overline{32}\,\overline{02}^2. \end{aligned}$$

If we look upon the form (4) as a pencil of cubics in t with coefficients $a + \tau b$ this sextic $\Delta_{a,b}$ should be expressible in terms of combinants of the pencil. The two linear combinants are * 5° and 6° or

$$\begin{aligned} (15)\qquad c_{1,4} &= 6\,\overline{01}t^4 + 12\,\overline{02}t^3 + 6[\overline{03} + 3\,\overline{12}]t^2 + 12\,\overline{13}t + 6\,\overline{23}, \\ c_{1,0} &= \overline{03} - 3\,\overline{12}. \end{aligned}$$

Denote the invariant of degree 3 of $c_{1,4}$ (its catalectic determinant) by $c_{3,0}$. Then

$$(16)\qquad \Delta_{a,b} = X_0 X_3 - X_1 X_2 = \tfrac{1}{54}(c_{3,0} + c^3_{1,0}).$$

---

* Cf. W. F. Shenton, this *Journal*, Vol. 37 (1915), p. 246.

We may note that the determinant $\Delta_{a,b}$ when expanded will not be formally equal to (14) since the identity

$$(17) \qquad \overline{01}\,\overline{23} + \overline{02}\,\overline{31} + \overline{03}\,\overline{12}$$

permits of modifying expressions of degree 2, 2 or higher in a, b. As a partial recapitulation we state

(18) *The general plane curve of genus* 4 *can be birationally transformed in two ways into the form* (14) *corresponding to the two g_1^3's upon it. This form depends upon the choice of four projective pencils $a_i + \tau b_i$ $(i = 0, 1, 2, 3)$ and involves* 8 *disposable constants for the centers of the pencils and* 9 *absolute constants for the projectivities.*

The complete system of combinant invariants of the pencil of cubics consists of $c_{1,0}$ and the invariants i, j of the binary quartic $c_{1,4}$. But we find that $i = 2(a_0a_4 - 4a_1a_3 + 3a_2^2) = 6c^2_{1,0}$ and that

$$j = 6 \begin{vmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \end{vmatrix} = 6c_{3,0}$$

where $a_0, \cdots, a_4$ are the coefficients of the quartic $c_{1,4}$. Thus all combinant invariants of the pencil are expressible in terms of $c_{1,0}$ and $c_{3,0}$. In particular the discriminant is $\frac{1}{27}(i^3 - 6j^2) = 8(c^6_{1,0} - c^2_{3,0}) = 8(c^3_{1,0} - c_{3,0})(c^3_{1,0} + c_{3,0})$. Now $c_{1,4}$, the jacobian of the pencil, has a double root when either some member of the pencil has a triple root and x is on $\Delta_{a,b}$, or the pencil has a fixed root and x is on $\bar{S}_2(t)$. We have already identified the factor $c^3_{1,0} + c_{3,0}$ of the discriminant whence the factor $c^3_{1,0} - c_{3,0}$ equated to zero is the rational sextic $\bar{S}_2(t)$. We observe that $\Delta_{a,b}$ and $\bar{S}_2(t)$ osculate at the 12 points where the conic $c_{1,0}$ cuts the sextic $c_{3,0}$ with an osculation tangent which belongs to $c_{3,0}$. At these points $i = j = 0$ and $c_{1,4}$ has a triple root. This occurs when the pencil of cubics has one member with a triple factor which is also a fixed factor i. e. at the flex points of perspective cubics. Of these there are precisely 12 given by the values τ for which $8°$ or $\bar{F}$ has a double root and they are the branch points of the function $t(\tau)$ on $\Delta_{a,b}$. Since a flex on a perspective cubic envelope is a contact of the cubic and $\bar{S}_2(t)$ the flex tangents are the osculation tangents of $\Delta_{a,b}$ and $\bar{S}_2(t)$. Hence

(19) *There are* 12 *perspective cubics with flex points at the meets of $c_{1,0}$ and $c_{3,0}$. The sextics $\Delta_{a,b} \equiv c^3_{1,0} + c_{3,0} = 0$ and $\bar{S}_2(t) \equiv c^3_{1,0} - c_{3,0} = 0$, osculate at these points with the flex tangents as common tangents. The* 12 *points are the branch points on $\Delta_{a,b}$ of the function $t(\tau)$ defined by $\bar{F} = 0$.*

In order to connect the figure of the perspective cubics of $\bar{S}_2(t)$ with the figure of two cubic curves in **4**, as well as to obtain the ternary equation of the covariant conic $K(\tau)$ which hitherto has been taken as a norm conic, we consider the matrix,

$$\left\|\begin{matrix} a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & b_3 \end{matrix}\right\| \tag{20}$$

(a_i and b_i linear in the ternary variable x),

which as observed in (18) defines the apparatus in the plane. We interpret a, b as points in space with reference to a norm cubic C_2 with parameter t which amounts practically to the choice of a coördinate system in space. Then the point $a + \tau b$ is on the three planes of C_2 furnished by the cubic (4) in t. Thus for given x and variable τ in $a + \tau b$ we have a line l_x in space whose points are named by the parameter τ. On the other hand for given τ and variable x in $a + \tau b$ we have a plane π_τ whose points are in projective correspondence with the points x of the plane of the given sextic $\bar{S}_2(t)$.

From the ternary identity between four lines we find for the coördinates π of the plane π_τ the values

$$\begin{aligned} \pi_0 &= (a_1a_2a_3) + \tau[(b_1a_2a_3) + (a_1b_2a_3) + (a_1a_2b_3)] + \\ &\qquad \tau^2[(a_1b_2b_3) + (b_1a_2b_3) + (b_1b_2a_3)] + \tau^3(b_1b_2b_3), \\ -\pi_1 &= (a_0a_2a_3) + \tau[(b_0a_2a_3) + (a_0b_2a_3) + (a_0a_2b_3)] + \\ &\qquad \tau^2[(a_0b_2b_3) + (b_0a_2b_3) + (b_0b_2a_3)] + \tau^3(b_0b_2b_3), \\ & \qquad \cdots\cdots\cdots\cdots \end{aligned} \tag{21}$$

Remembering that plane t of C_2 has been used in (4) as

$$\eta_0 = t^3, \quad \eta_1 = 3t^2, \quad \eta_2 = 3t, \quad \eta_3 = 1$$

whence point t of C_2 is

$$y_0 = 1, \quad y_1 = -t, \quad y_2 = t^2, \quad y_3 = -t^3$$

we find that the condition that the plane π_τ be incident with point t of C_2 is precisely the form 8° whence the planes π_τ are the planes of the cubic curve C_1. If now we ask for planes π_τ on the point $a + \tau' b$ we find that in the incidence condition the constant term and the term in $\tau' t^3$ vanish while the pairs of coefficients of τ, τ'; $\tau^2, \tau\tau'$; τ^3; $\tau^2\tau'$ differ in sign due entirely to the four-term ternary identities. Hence the incidence condition contains the factor $(\tau - \tau')$ and another factor linear in x and of the second degree in τ. Thus of the three planes π_τ on $a + \tau' b$ two are independent of τ' and therefore are on the

axis l_x of C_1 and the remaining plane τ' is on the point $a+\tau' b$ of this axis. The two planes of C_1 on the axis l_x are furnished by the equation

$$
\begin{aligned}
&[b_0(a_1a_2a_3)-b_1(a_0a_2a_3)+b_2(a_0a_1a_3)-b_3(a_0a_1a_2)] \\
&+\tau[b_0\{(b_1a_2a_3)+(a_1b_2a_3)+(a_1a_2b_3)\}-b_1\{(b_0a_2a_3)+(a_0b_2a_3)+(a_0a_2b_3)\} \\
(22)\quad &+b_2\{(b_0a_1a_3)+(a_0b_1a_3)+(a_0a_1b_3)\}-b_3\{(b_0a_1a_2)+(a_0b_1a_2)+(a_0a_1b_2)\}] \\
&+\tau^2[-a_0(b_1b_2b_3)+a_1(b_0b_2b_3)-a_2(b_0b_1b_3)+a_3(b_0b_1b_2)].
\end{aligned}
$$

(23) *The point $a+\tau b$ for variable x and fixed τ describes a plane π_τ of C_1 whose coördinates are given in* (21); *for variable τ and fixed x, an axis l_x of C_1 whose two planes are given by* (22). *To the points x on the sextic cusp locus $\Delta_{a,b}$ there correspond the axes l_x of C_1 on points of C_2 and to the nodes of $\Delta_{a,b}$ there correspond the six axes of C_1 which are bisecants of C_2; to the points x on $\bar{S}_2(t)$ there correspond the axes of C_1 on planes of C_2 and to the nodes of $\bar{S}_2(t)$, the ten common axes of C_1, C_2.*

It is clear that (22) is the parametric line equation of the conic $K(\tau)$ of section 4. The symbolic form of this conic is obtained by taking the apolarity condition of the binary cubic $(\pi'''x)(a'''\tau')(a'''t)^3$ as to the form $8°$ and removing the factor $(\tau\tau')$ by operating with $(\tau\tau')$ to produce

$$(24)\quad (\pi\pi'\pi'')(\pi'''x)(a'\tau)(aa')^3(a''a''')^3\{2(aa''')(a''\tau)+(a''a''')(a\tau)\}=0.$$

(25) *The conic $K(\tau)$ on which the nodes of $\bar{S}_2(t)$ determine the nodal parameters of the paired sextic $\bar{S}_1(\tau)$ has for parametric equation the symbolic and non-symbolic forms* (24) *and* (22) *respectively.*

If we pass back from the general form (4) to the particular forms (2), (3) for which $K(\tau)$ is isolated as a norm conic then we have immediately or by the use of a somewhat noteworthy matrix property that

$$
\begin{aligned}
&(a_1a_2a_3)=R_{00}, \qquad (b_1a_2a_3)+(a_1b_2a_3)+(a_1a_2b_3)=-R_{10}, \\
&(a_0a_2a_3)=-R_{01}, \quad (b_0a_2a_3)+(a_0b_2a_3)+(a_0a_2b_3)=R_{11}, \\
&\cdots\cdots\cdots\cdots \\
&(a_1b_2b_3)+(b_1a_2b_3)+(b_1b_2a_3)=R_{20}, \qquad (b_1b_2b_3)=-R_{30}, \\
(26)\quad &(a_0b_2b_3)+(b_0a_2b_3)+(b_0b_2a_3)=-R_{21}, \quad (b_0b_2b_3)=R_{31}, \\
&\cdots\cdots\cdots\cdots
\end{aligned}
$$

This verifies again that the form $8°$ is the form F, the incidence condition of plane τ of C_1 and point t of C_2. Moreover the line conic (22) reduces to $R(x_0-x_1\tau+x_2\tau^2)$ which is the normal form of $K(\tau)$ used in (2). Hence

(27) *The invariant R of degree 4 of the form F appears in the form R^3 of degree 12 in the coefficients of the $\left(\begin{smallmatrix}1 & 1 & 3\\ 2 & 1 & 1\end{smallmatrix}\right)$ form* (4), *the discriminant of the conic* (22).

Other comitants of the form (4) are easily interpretable with reference to the curves C_1, C_2. Thus 5° furnishes the four parameters t of tangents of C_2 which are cut by the axis l_x of C_1; 6° is the conic for which the axis l_x of C_1 is in the null system of C_2; and 7° furnishes for point τ on the axis l_x of C_1 the parameters t of the bisecant of C_2 on this point.

6. The Form $(\frac{1}{3}\frac{1}{2}\frac{1}{2})$. The Cubic Surface with Isolated Double-Six.

This section is preliminary to the next in which the projectively general plane sextic of genus 4 with 13 absolute constants is introduced. We consider the form,

$$(hz)(rx)(sy) = 0, \tag{1}$$

where z is a point in S_3, x a point in S_2, and y a point in S'_2. This form has 36 coefficients and therefore $35 - 15 - 8 - 8 = 4$ absolute projective constants. For given x, y we have in (1) a plane in S_3. However this plane is indeterminate for six pairs of points x, y say p_i, q_i $(i = 1, \cdots, 6)$, which are the six solutions of four bilinear equations in x, y and therefore are associated six-points. These six-points uniquely determine a Cremona transformation T of the fifth order between the planes S_x and S_y, for which the six-points are double F-points.* A given plane u on points z_1, z_2, z_3 will be obtained in (1) from the ∞^1 pairs x, y which satisfy the three bilinear relations, $(hz_i)(rx)(sy) = 0$ $(i = 1, 2, 3)$. The points x and points y of these ∞^1 pairs lie respectively on the two cubic curves in S_x, S_y

$$(hh'h''u)(ss's'')(rx)(r'x)(r''x) = 0, \tag{2}$$

$$(hh'h''u)(rr'r'')(sy)(s''y) = 0, \tag{3}$$

which pass respectively through the six-points p_i, q_i.

For given x and variable y in (1) we have a net of planes on a point z in space whose equation is (2). Hence the equation (2) furnishes the mapping of the plane S_x upon a cubic surface C^3 by means of the web of cubic curves on the six-point p. The equation of C^3, obtained by eliminating x from the equations, $(hz)(rx)s_i = 0$ $(i = 0, 1, 2)$, is

$$(hz)(h'z)(h''z)(rr'r'')(ss's'') = 0. \tag{4}$$

From the symmetry of (4) in r, s we see that C^3 is also the map of the plane S_y by the cubic curves (3) on the six-point q. Due to the well known prop-

* Cf. Coble, *Trans. Amer. Math. Soc.*, Vol. 9 (1908), p. 398; and Vol. 16 (1915), §§ 1, 2.

erties of T we conclude that the points p map by (2) into a line-six on C^3 and the points q by (3) into the cross line-six on C^3 and corresponding pairs x, y of T map into the same point of C^3. Under T then the cubic curves (2), (3) for given u correspond.

The equation of the transformation T is easily obtained. If in (1) z is a point on C^3 then according to (4) the resulting correlation is singular and the singular points x, y in S_x, S_μ are a pair of co-points of T. The dual form of the correlation is then the product $(x\xi).(y\eta)$ of these singular points. Hence given x in (2) we have a point of C^3 which substituted in (1) gives rise to $(hh'h''h''')(ss's'')(rx)(r'x)(r''x)(r'''x')(s'''y')=0$, a degenerate correlation in x', y' whose dual form is

$$(hh'h''h''')(h^{IV}h^{V}h^{VI}h^{VII})(ss's'')(s^{IV}s^{V}s^{VI})(rx)(r'x)(r''x)(r^{IV}x)$$
$$(r^{V}x)(r^{VI}x)(r'''r^{VII}\xi)(s'''s^{VII}\eta).$$

If this is the product $(x\xi).(y\eta)$ the result of operating upon it with $(x\xi)$ is $(y\eta)=0$ the equation of the co-point of x under T. This result is

$$(5)\qquad (hh'h''h''')(h^{IV}h^{V}h^{VI}h^{VII})(ss's'')(s^{IV}s^{V}s^{VI})$$
$$(rr'''r^{VII})(r'x)(r''x)(r^{IV}x)(r^{V}x)(r^{VI}x)(s'''s^{VII}\eta)=0.$$

If z_1 is a point on C^3, the map of the pair of co-points x, y of T, the dual form of the correlation (1) for $z=z_1$ is $(hz_1)(h'z_1)(rr'\xi)(ss'\eta)=\lambda.(x_1\xi).(y_1\eta)$. If we operate with this degenerate form on (1), which amounts to substituting the co-pair x_1, y_1 in (1) we obtain

$$(hz_1)(h'z_1)(h''z)(rr'r'')(ss's'')=0$$

which is the tangent plane of C^3 at the point z_1. Hence the co-pairs x, y, of T determine in (1) the tangent planes of C^3. If for given x we use the equation (5) for the co-point y of x then the equation of the tangent plane of C^3 at the point z which corresponds to x is

$$(6)\qquad (h^{VIII}z)(hh'h''h''')(h^{IV}h^{V}h^{VI}h^{VII})(ss's'')(s^{IV}s^{V}s^{VI})(s'''s^{VII}s^{VIII})$$
$$(rr'''r^{VII})(r'x)(r''x)(r^{IV}x)(r^{V}x)(r^{VI}x)(r^{VIII}x)=0.$$

Hence in (6) we have the map of the plane upon the surface C^3, as an envelope of class 12, by means of a web of sextic curves with nodes at the six-points p. Two curves of this web have 12 variable intersections.

Clearly the whole apparatus above is determined by one six-point, say p_i in S_x. The advantage in beginning with the form (1) lies in the fact that the webs (2) and (3) are necessarily on associated six-points. Indeed the present scheme of defining a cubic surface and a double-six on it is as flexible as can be expected.

7. The Form $\binom{1\,1\,1\,1}{1\,1\,2\,2}$. The Projectively General Plane Sextic of Genus 4.

Into the space figure just described in **6** we inject a proper quadric Q with generators t, τ. If by proper choice of the coördinate system in space we take the quaternary, and the parametric, equations in points and planes of Q to be

$$(1)\quad Q:\qquad \begin{array}{llll} z_0z_3 - z_1z_2 = 0; & \zeta_0\zeta_3 - \zeta_1\zeta_2 = 0; & & \\ z_0 = -\tau, & z_1 = t\tau, & z_2 = 1, & z_3 = -t; \\ \zeta_0 = t, & \zeta_1 = 1, & \zeta_2 = \tau t, & \zeta_3 = \tau; \end{array}$$

then a given point z or a given plane ζ is determined by a bilinear form in τ, t:

$$(2)\qquad z_0t + z_1 + z_2t\tau + z_3t = 0, \quad \text{or} \quad -\zeta_0\tau + \zeta_1t\tau + \zeta_2 - \zeta_3t = 0$$

respectively. A quaternary form of order k in z or ζ is polarized k times and for each of the k sets of variables a set of variables t, τ is introduced as in (1). Thus the quaternary symbolic notation is replaced by a double binary notation. A given bilinear form, compared with (2), determines a point z and a plane ζ which are pole and polar as to Q. The four bilinear forms $(a\tau)(\alpha t)$, $(b\tau)(\beta t)$, $(c\tau)(\gamma t)$, $(d\tau)(\delta t)$ determine 4 points on a plane or 4 planes on a point if

$$(3)\begin{vmatrix} (bc)(ad) & (ca)(bd) \\ (\beta\gamma)(\alpha\delta) & (\gamma\alpha)(\beta\delta) \end{vmatrix} = \begin{vmatrix} (ca)(bd) & (ab)(cd) \\ (\gamma\alpha)(\beta\delta) & (\alpha\beta)(\gamma\delta) \end{vmatrix} = \begin{vmatrix} (ab)(cd) & (bc)(ad) \\ (\alpha\beta)(\gamma\delta) & (\beta\gamma)(\alpha\delta) \end{vmatrix} = 0,$$

A point or plane determined by the bilinear form $(a\tau)(\alpha t) = 0$ is on the quadric if

$$(4)\qquad (aa')(\alpha\alpha') = 0$$

A line on two points or two planes determined by two bilinear forms, $(a\tau)(\alpha t)$, $(b\tau)(\beta t)$ touches the quadric if

$$(5)\qquad (aa')(\alpha\alpha').(bb')(\beta\beta') - [(ab)(\alpha\beta)]^2 = 0.$$

If now in the form **6**(1) we replace the coördinates z as in (1) we obtain a form $\binom{1\,1\,1\,1}{1\,1\,2\,2}$.

$$(6)\qquad (k\tau)(\kappa t)(rx)(sy)$$

with $2 \cdot 2 \cdot 3 \cdot 3 = 36$ coefficients and $35 - 3 - 3 - 8 - 8 = 13$ absolute constants, which arise from the four absolute constants of the cubic surface C^3 of **6** and the nine for the quadric Q inserted here. The quadric meets the cubic surface in a normal curve of genus 4 so that we have in space the figure of the normal sextic and a particular one of the ∞^4 cubic surfaces through it and upon this surface a particular double six. In the planes S_x, S_y we have

projectively general (with thirteen absolute constants) sextics of genus 4 with nodes at p_i and q_i respectively, transforms of each other under T. We may therefore regard the form (6) as a definition of the projectively general plane sextic of genus 4.

The equation of the cubic surface C^3 in terms of (6) is

$$(7) \qquad (k\tau)(k'\tau')(k''\tau'')(\kappa t)(\kappa' t')(\kappa'' t'')(rr'r'')(ss's'') = 0$$

while the equation of the normal sextic on the quadric is obtained by setting $\tau, t = \tau', t' = \tau'', t''$ in (7). The canonical adjoints of the two plane sextics in S_x, S_y are obtained by substituting for u in **6**(2), (3) the proper bilinear expressions in t, τ; thus we have according to (3)

$$(8) \quad \{(k'k'')(k\tau)(\kappa''\kappa)(\kappa' t) - (k''k)(k'\tau)(\kappa'\kappa'')(\kappa t)\} \, (ss's'')(rx)(r'x)(r''x) = 0,$$

$$(9) \quad \{ \qquad \text{"} \qquad \} \, (rr'r'')(sy)(s'y)(s''y) = 0.$$

For fixed τ and variable t in (6) we have a pencil of correlations in which there are three degenerate members whose singular points $(x\xi).(y\eta)$ form a τ-triad of the sextics. On such a τ-triad and the points p_i or q_i there is a pencil of adjoint cubics given by variable t in (8) or (9).

For given x, y in (6) the plane in S_3 determined by the bilinear form is, according to (4), a plane of Q if

$$(10) \qquad (cx)^2(dy)^2 = (kk')(\kappa\kappa')(rx)(r'x)(sy)(s'y) = 0.$$

For given x and variable y in (6) the planes all pass through the point z of C^3, which is the map of x; thus for given x in (10) we have a conic on S_y, the locus of points y which with x determine in (6) the planes of the quadric cone with vertex at z and tangent to Q. If z is on Q this quadric cone consists of two pencils whence y in (10) runs over two lines. Hence the equations of the sextic curves of genus 4 in S_x, S_y which map into the curve C^3, Q are

$$(11) \quad (dd'd'')^2(cx)^2(c'x)^2(c''x)^2 = 0, \quad (cc'c'')^2(dy)^2(d'y)^2(d''y)^2 = 0.$$

Before discussing other comitants of (6) and (10) we recall certain birational properties of a ternary 6-nodal sextic σ_x of genus 4. If the 6 nodes p_i are on a conic the linear series $I_2{}^6$ cut out by the line sections is a special and incomplete involution being contained in the unique canonical involution $C = I_3{}^6$. The sextic is then the projection of a normal space sextic and the plane of σ_x is mapped by cubics on p_i into a cubic surface with a node at the

center of projection. If the 6 nodes are not on a conic the I_2^6 of line sections is complete and σ_x may be so birationally transformed into a similar sextic that any I_2^6 becomes the line sections. The complete I_2^6's occur in pairs such that

$$I_2^6 + I'^6_2 = 2C$$

a division into pairs invariant under birational transformation. On the normal space curve quadrics on a set of I_2^6 cut out I'^6_2 and vice versa. In the plane if I_2^6 is the line sections of σ_x then I'^6_2, being cut out by quintics with nodes at p_i, is the line sections of a sextic σ_y with nodes at q_i the transform of σ_x under T. Just as every I_2^6 determines another I'^6_2 coresidual with it in $2C$, so every I_2^6 determines an I_8^{12} coresidual with it in $3C$. Thus if

$$I_2^6 + I_8^{12} = 3C, \quad I'^6_2 + I'^{12}_8 = 3C$$

then I'^{12}_8 is the involution cut out on σ_x by its quartic adjoints, and I_8^{12} that similarly cut out on σ_y.

Any I_2^6 has a covariant I_8^{12} such that $2I_2^6 = I_8^{12}$. If q_2^6 is an involution of the contacts of one of the 255 systems of contact quadrics i. e. if $2q_2^6 = 2C$, or if $C - q_2^6$ is a virtual half period set then also $I_8^{12} = 2(I_2^6 + C - q_2^6$. If I_8^{12} is an involution of quartic adjoints it appears therefore that there are 256 systems of contact adjoint quartics. If for I'^{12}_8 of the preceding paragraph as the adjoint quartics of σ_x we have a system J'^6_2 of contact quartics then it is easily verified from the above that the involution J_2^6 such that $J_2^6 + J'^6_2 = 2C$ has the property $2J_2^6 = I_8^{12}$, and is a contact system of quartic adjoints of σ_y.

We now consider the dual form,

$$(12) \qquad (k\tau)(k'\tau)(\kappa t)(\kappa' t)(rr'\xi)(ss'\eta) = 0$$

of the correlation (6) between x and y. If τ, t is a point of Q on the space sextic, (12) becomes a product $(x\xi).(y\eta)$ of the co-points x, y under T on σ_x, σ_y respectively. Thus for given ξ and variable η we have in (12) a net of quartic curves on Q cut out by the quadrics on the 6 points of the space sextic which correspond to the line section of σ_x by ξ. Thus the form (12) determines the quadrics on the sets of the I_2^6 and I'^6_2 which correspond to line sections of σ_x and σ_y. Conversely given a set of I_2^6 on the space sextic, the quadrics on it which cut out I'^6_2 contain a parameter η linearly and for fixed η and therefore fixed set of I'^6_2 they also contain a parameter ξ linearly and thereby a form (12) is defined. The dual form of this (12) is again the form (6), to within the (3, 3) form in τ, t which determines the space sextic, and again the sextics are obtained as in (11).

We now consider the forms

$$(13) \qquad (cx)^2(c'x)^2(dd'\eta)^2 = 0, \qquad (dy)^2(d'y)^2(cc'\xi)^2 = 0.$$

For given x on σ_x, $(cx)^2(dy)^2 = 0$ is a pair of lines in S_y which meet in a point y which with x determines in (6) the tangent plane to Q at the point of the space sextic which corresponds to x; whence (13) is the equation $(\eta y)^2 = 0$ of the square of this point. A given conic in S_y, $(ey)^2 = 0$, contains 24 of these points y since the quartic (not an adjoint) $(cx)^2(c'x)^2(dd'e)^2 = 0$ meets an σ_x in 24 points whence the locus of such points y is a 12-ic curve birationally equivalent to σ_x. If $(ey)^2 = 0$ is a line pair $(\eta y) . (\eta' y) = 0$ the quartic $(cx)^2(c'x)^2(dd'\eta)(dd'\eta') = 0$ meets σ_x in the two sets of 12 points which correspond to the meets of η, η' with the 12-ic. Thus for variable η' we have an involution I_2^{12} cut out on σ_x which maps σ_x upon the 12-ic and for $\eta = \eta'$ we have a contact system of non-adjoint quartics of the sextic σ_x. A system of contact adjoint quartics will be obtained in **9**(9).

We now make an application of the preceding remarks to the sextic, $\Delta_{a,b}$, the cusp locus of the perspective cubics of $\bar{S}_2(t)$. If we polarize the form $F = (a\tau)^3(\alpha t)^3$ into

$$(14) \qquad (a\tau_1)(a\tau_2)(a\tau)(\alpha t)(\alpha t_1)(\alpha t_2)$$

and replace the pairs τ_1, τ_2 and t_1, t_2 by points x and y referred in Darboux coördinates to the conics $K(\tau)$, $K(t)$ in S_x, S_μ respectively, we have a form

$$(15) \qquad (a\tau)(\alpha t)(\pi x)(\rho y)$$

of the type (6). For the space cubic curves C_1, C_2 the form (15) is the incidence condition of the point τ on the axis l_x of C_1 and the plane t on the bisecant λ_y of C_2. For x a point on $\Delta_{a,b}$ the form $(\pi x)(a\tau)(\alpha t)^3$ is a perfect cube in t and the form $(\pi x)(a\tau)(\alpha t)(\rho y)$ vanishes identically in y whence x is a singular point of the correlation (15) for the given τ, t or x is a point of the sextic σ_x. The bisecant l_x of C_1 is then on the point t of C_2. Referring then to **5**(23) we may state the theorem

(16) *The cusp loci of perspective cubics of the diagonal rational sextics $\bar{S}_2(t)$, $S_1(\tau)$ are not merely birationally equivalent to each other and to F but they are transforms of each other under a quintic Cremona transformation T. The six axes of C_1 which are bisecants of C_2 determine six pairs of parameters on either curve which when plotted with reference to norm conics yield two associated six-points.*

Naturally on σ_x the cusp triads are τ-triads whereas on σ_y the cusp triads

are t-triads. We shall prove in **9**(1) that on σ_x the t-triads form triangles whose sides touch $K(\tau)$ and on σ_y the τ-triads behave similarly with respect to $K(t)$.

The parametric equation of the space sextic on Q which is the normal form of the two cusp loci is according to (7) $\bar{F}=0$. The two I_2^{6}'s on $\bar{F}$ which represent the line sections of the cusp loci are determined from the form (12) which in the double binary notation (ξ, η referred to $K(\tau)$, $K(t)$ respectively) becomes

$$(17) \qquad (aa')(\alpha\alpha')(a\tau)(a'\tau)(\alpha t)(\alpha' t)(a\tau_1)(a'\tau_1)(\alpha t_1)(\alpha' t_1).$$

Thus we see that

(18) *If* $(c\tau)^2$ *is a line section of* $K(\tau)$, *the three quartic space curves on* Q *obtained by setting the coefficients of* t_1 *in* $(aa')(\alpha\alpha')(a\tau)(a'\tau)(\alpha t)(\alpha' t)(ac)$ $(a'c)(\alpha t_1)(\alpha' t_1)$ *equal to zero meet in six points of* $\bar{F}=0$ *which correspond to this line section of the cusp locus of* $\bar{S}_2(t)$.

The line sections on $\bar{F}$ of the cusp locus of $S_1(\tau)$ are of course similarly obtained. We shall find in the next section another interpretation of the form (17).

We have obtained in **5**(10), the equation of the cusp locus $\Delta_{a,b}=\sigma_x$ as a three-row determinant whose minors are adjoint quartics. On the other hand in (11) we have the equation of σ_x as a symmetric three-row determinant, the discriminant of the conic (10). To obtain this form of the equation of $\Delta_{a,b}$ we take the form **7**(4) and polarize it twice as to t_1 obtaining

$$(\tau t a_0+tb_0+\tau a_1+b_1)t_1^2+(\tau t a_1+tb_1+\tau a_2+b_2)2t_1 \\ +(\tau t a_2+tb_2+\tau a_3+b_3)$$

in which $t_1^2 : 2t_1 : 1 = y_0 : y_1 : y_2$. We then take the bilinear invariant in τ, t of this form with itself to obtain (10) and write the discriminant (11) which is, in the notation of **5**(13),

$$(19) \qquad 2\Delta_{a,b}=\begin{vmatrix} 2\,\overline{01} & \overline{02} & \overline{03}-\overline{12} \\ \overline{02} & 2\,\overline{12} & \overline{13} \\ \overline{03}-\overline{12} & \overline{13} & 2\,\overline{23} \end{vmatrix}.$$

The particular case of the form (6) for which the variables x, y are cogredient and the form is merely the polar of $(k\tau)(\kappa t)(rx)^2$ is especially important in connection with Wirtinger's form of the plane sextic of genus four and will be discussed later.

8. The Fundamental Combinants $\Gamma, \bar{\Gamma}$ of $F, \bar{F}$.

We have had occasion in connection with the nodes of the cusp locus $\Delta_{a,b}$ to consider such pencils of cubics in t as are determined by the members $(a\tau_1)^3(at)^3$, $(a\tau_2)^3(at)^3$ $(\tau_1 \neq \tau_2)$. If t_1, t_2 belong to the same member of the pencil then the fundamental combinant of Gordan for the pencil is

$$(1) \qquad \Gamma = \begin{vmatrix} (a\tau_1)^3(at_1)^3 & (a\tau_2)^3(at_1)^3 \\ (a\tau_1)^3(at_2)^3 & (a\tau_2)^3(at_2)^3 \end{vmatrix} .$$

Evidently $\Gamma = 0$ expresses also that τ_1, τ_2 belong to the same member of the pencil determined by the members $(a\tau)^3(at_1)^3$, $(a\tau)^3(at_2)^3$. We call Γ the fundamental combinant of $F = (a\tau)^3(at)^3$ and $\bar{\Gamma}$ the corresponding fundamental combinant of $\bar{F} = (\bar{a}\tau)^3(\bar{a}t)^3$. If we disregard in Γ, $\bar{\Gamma}$ the factors $(t_1t_2)(\tau_1\tau_2)$ then their interpretations with respect to the space cubics C_1, C_2 is that Γ vanishes when bisecant λ_x $(x = \tau_1, \tau_2)$ of C_1 meets axis l_y $(y = t_1, t_2)$ of C_2; whereas $\bar{\Gamma}$ vanishes when axis l_x of C_1 meets bisecant λ_y of C_2.

Since Γ, and therefore $\bar{\Gamma}$ also, is expressible in terms of the two-row minors of R we shall proceed to find their values in terms of polars of the four comitants of **3**(12). On expanding (1) we have

$$\begin{aligned}
\Gamma &= (a\tau_1)^3(a'\tau_2)^3[(at_1)^3(a't_2)^3 - (a't_2)^3(a't_1)^3] \\
&= \tfrac{1}{2}[(a\tau_1)^3(a'\tau_2)^3 - (a\tau_2)^3(a'\tau_1)^3][(at_1)^3(a't_2)^3 - (at_2)^3(a't_1)^3] \\
&= \tfrac{1}{2}(\tau_1\tau_2)(t_1t_2).(aa')(aa') \\
&\qquad \{(a\tau_1)^2(a'\tau_2)^2 + (a\tau_1)(a\tau_2)(a'\tau_1)(a'\tau_2) + (a\tau_2)^2(a'\tau_1)^2\} \\
&\qquad \{(at_1)^2(a't_2)^2 + (at_1)(at_2)(a't_1)(a't_2) + (at_2)^2(a't_1)^2\} \\
&= (\tau_1\tau_2).(t_1t_2)\{c_1 + c_2 + c_3 + c_4 + \tfrac{1}{2}c_5\}
\end{aligned}$$

(2) where

$$\begin{aligned}
c_1 &= (aa')(aa')(a\tau_1)^2(a'\tau_2)^2(at_1)^2(a't_2)^2 \\
c_2 &= (aa')(aa')(a\tau_1)^2(a'\tau_2)^2(at_1)(at_2)(a't_1)(a't_2). \\
c_3 &= (aa')(aa')(a\tau_1)^2(a'\tau_2)^2(at_2)^2(a't_1)^2 \\
c_4 &= (aa')(aa')(a\tau_1)(a\tau_2)(a'\tau_1)(a'\tau_2)(at_1)^2(a't_2)^2 \\
c_5 &= (aa')(aa')(a\tau_1)(a\tau_2)(a'\tau_1)(a'\tau_2)(at_1)(at_2)(a't_1)(a't_2).
\end{aligned}$$

These same five terms c occur in five other expression namely $(b\tau_1)^2(b\tau_2)^2$ $(\beta t_1)^2(\beta t_2)^2$, $(t_1t_2)^2.(c\tau_1)^2(c\tau_2)^2$, $(\tau_1\tau_2)^2.(\gamma t_1)^2(\gamma t_2)^2$, $(t_1t_2)^2.(\tau_1\tau_2)^2.\delta$, $(t_1t_2).(\tau_1\tau_2).(e\tau_1)(e\tau_2)(\epsilon t_1)(\epsilon t_2)$ where $(e\tau)^2(\epsilon t)^2 \equiv (aa')^2(aa')^2(a\tau)(a'\tau)$ $(at)(a't)$. On forming these five polars and products and expressing them in terms of the five c's we have

$$
(3)\quad
\begin{aligned}
4K &\equiv 4(b\tau_1)^2(b\tau_2)^2(\beta t_1)^2(\beta t_2)^2 = c_1 + 4c_2 + c_3 + 4c_4 + 8c_5,\\
4L &\equiv 4(t_1t_2)^2.(c\tau_1)^2(c\tau_2)^2 \qquad = c_1 - 2c_2 + c_3 + 4c_4 - 4c_5,\\
4M &\equiv 4(\tau_1\tau_2)^2.(\gamma t_1)^2(\gamma t_2)^2 \qquad = c_1 + 4c_2 + c_3 - 2c_4 - 4c_5,\\
4N &\equiv 4(\tau_1\tau_2)^2.(t_1t_2)^2.\delta \qquad = c_1 - 2c_2 + c_3 - 2c_4 + 2c_5,\\
4U &\equiv 2(t_1t_2).(\tau_1\tau_2).(e\tau_1)(e\tau_2)(\epsilon t_1)(\epsilon t_2) = c_1 \qquad - c_3
\end{aligned}
$$

On solving these for the forms c we have

$$
(4)\quad
\begin{aligned}
9c_1/2 &= K + 2L + 2M + 4N + U,\\
9c_3/2 &= K + 2L + 2M + 4N - U,\\
9c_2/2 &= K - L + 2M - 2N,\\
9c_4/2 &= K + 2L - M - 2N,\\
9c_5/2 &= K - L - M + N.
\end{aligned}
$$

Setting these values in the expression for Γ we have

$$
(5)\quad
\begin{aligned}
\Gamma &= (\tau_1\tau_2).(t_1t_2).\{(b\tau_1)^2(b\tau_2)^2(\beta t_1)^2(\beta t_2)^2 + (t_1t_2)^2.(c\tau_1)^2(c\tau_2)^2\\
&\qquad + (\tau_1\tau_2)^2.(\gamma t_1)^2(\gamma t_2)^2 + (\tau_1\tau_2)^2.(t_1t_2)^2.\delta\}\\
&= (\tau_1\tau_2).(t_1t_2).\{K + L + M + N\}.
\end{aligned}
$$

From the non-symbolic forms of the four comitants in **3**(16) we readily find with reference to **3**(6), (7), (8), (9) that these comitants formed for $\bar{F}$ are reproduced multiplied by the factor $R/9$ except for $(c\tau)^4$, $(\gamma t)^4$ which also are changed in sign.

Hence we have

$$
(6)\qquad \bar{\Gamma} = (R/9).(\tau_1\tau_2)(.t_1t_2).\{K - L - M + N\}.
$$

From the value of the form c_5 in (4) we have also

$$
(7)\quad \bar{\Gamma} = (R/2).(\tau_1\tau_2).(t_1t_2).(aa')(aa')(a\tau_1)(a'\tau_1)(a\tau_2)(a'\tau_2)\\
(at_1)(a't_1)(at_2)(a't_2).
$$

This is the form **7**(17). From this we find that

$$
(8)\quad \Gamma = (8{}^1\!/_2 R).(\tau_1\tau_2).(t_1t_2).(\bar{a}\bar{a}')(\bar{a}\bar{a}')(\bar{a}\tau_1)(\bar{a}'\tau_1)(\bar{a}\tau_2)(\bar{a}'\tau_2)\\
(\bar{a}t_1)(\bar{a}'t_1)(\bar{a}t_2)(\bar{a}'t_2).
$$

We have obtained the locus of cusps of perspective cubics of $\bar{S}_2(t)$ from the pencil of cubics in t with members $(\pi x)(a\tau)(at)^3$, $(\pi x)(a\tau')(at)^3$ $(x = \tau_1, \tau_2)$. If t_1, t_2 belong to the same cubic of this pencil, the fundamental combinant of the pencil, to within the factors $(\tau\tau')(t_1t_2)$, is

$$
\begin{aligned}
&3(aa')(aa')(a\tau_1)(a'\tau_1)(a\tau_2)(a'\tau_2)\\
&\qquad\{(at_1)^2(a't_2)^2 + (at_1)(a't_1)(at_2)(a't_2) + (a't_1)^2(at_2)^2\}\\
&= 3(2c_4 + c_5) = 2(K + L - M - N).
\end{aligned}
$$

Hence

$$(9)\quad 3\begin{vmatrix}(\pi x)(a\tau)(at_1)^3 & (\pi x)(a\tau)(at_2)^3\\ (\pi x)(a\tau')(at_1)^3 & (\pi x)(a\tau')(at_2)^3\end{vmatrix}_{(x=\tau_1,\,\tau_2)} = (\tau\tau').(t_1t_2)\{K+L-M-N\}.$$

From this we conclude as above that

$$(10)\quad 3\begin{vmatrix}(\bar\pi x)(\bar a\tau)(\bar a t_1)^3 & (\bar\pi x)(\bar a\tau)(\bar a t_2)^3\\ (\bar\pi x)(\bar a\tau')(\bar a t_1)^3 & (\bar\pi x)(\bar a\tau')(\bar a t_2)^3\end{vmatrix}_{(x=\tau_1,\,\tau_2)} = (R/9).(\tau\tau').(t_1t_2)\{K-L+M-N\}.$$

When we form the corresponding pencils for the point $y = t_1, t_2$ in the plane S_y of the conic $K(t)$ we find that

$$(9')\quad 3\begin{vmatrix}(a\tau_1)^3(at)(\rho y) & (a\tau_2)^3(at)(\rho y)\\ (a\tau_1)^3(at')(\rho y) & (a\tau_2)^3(at')(\rho y)\end{vmatrix}_{(y=t_1,\,t_2)} = (tt').(\tau_1\tau_2)\{K-L+M-N\}.$$

$$(10')\quad 3\begin{vmatrix}(\bar a\tau_1)^3(\bar a t)(\bar\rho y) & (\bar a\tau_2)^3(\bar a t)(\bar\rho y)\\ (\bar a\tau_1)^3(\bar a t')(\bar\rho y) & (\bar a\tau_2)^3(\bar a t')(\bar\rho y)\end{vmatrix}_{(y=t_1,\,t_2)} = (R/9).(tt').(\tau_1\tau_2)\{K+L-M-N\}.$$

In (4) we have the values of the terms of the double second polars of $(b\tau)^4(\beta t)^4$. It is convenient to have the values also of the terms of the single second polars of this form and also of the second polars of $(c\tau)^4$ and $(\gamma t)^4$. These are collected here:

$$(11)\quad \begin{aligned}
&9(aa')(\alpha\alpha')(at)^2(a't)^2(a\tau)^2(a'\tau_1)^2\\
&\qquad = 2(b\tau)^2(b\tau_1)^2(\beta t)^4 + 4(\tau\tau_1)^2.(\gamma t)^4,\\
&9(aa')(\alpha\alpha')(at)^2(a't)^2(a\tau)(a'\tau)(a\tau_1)(a'\tau_1)\\
&\qquad = 2(b\tau)^2(b\tau_1)^2(\beta t)^4 - 2(\tau\tau_1)^2.(\gamma t)^4,\\
&9(aa')(\alpha\alpha')(a\tau)^2(a'\tau)^2(at)^2(a't_1)^2\\
&\qquad = 2(b\tau)^4(\beta t)^2(\beta t_1)^2 + 4(tt_1)^2.(c\tau)^4,\\
&9(aa')(\alpha\alpha')(a\tau)^2(a'\tau)^2(at)(a't)(at_1)(a't_1)\\
&\qquad = 2(b\tau)^4(\beta t)^2(\beta t_1)^2 - 2(tt_1)^2.(c\tau)^4,\\
&3(aa')^3(\alpha\alpha')(at)^2(a't_1)^2 = 4(\gamma t)^2(\gamma t_1)^2 + 8(tt_1)^2.\delta,\\
&3(aa')^3(\alpha\alpha')(at)(a't)(at_1)(a't_1) = 4(\gamma t)^2(\gamma t_1)^2 - 4(tt_1)^2.\delta,\\
&3(aa')(\alpha\alpha')^3(a\tau)^2(a'\tau_1)^2 = 4(c\tau)^2(c\tau_1)^2 + 8(\tau\tau_1)^2.\delta,\\
&3(aa')(\alpha\alpha')^3(a\tau)^2(a'\tau)(a\tau_1)(a'\tau_1) = 4(c\tau)^2(c\tau_1)^2 - 4(\tau\tau_1)^2.\delta.
\end{aligned}$$

The corresponding formulae for $\bar a\bar\alpha$ are easily supplied as above.

Four of the comitants of the second degree of the form $(\pi x)(a\tau)(at)^3$ which are listed in **5**, can be expressed in terms of the four self dual comitants when x is τ_1, τ_2. These are, from (11),

$$(1^2)\quad \begin{aligned} 3^\circ &\equiv 9(aa')(\alpha\alpha')(a\tau_1)(a'\tau_2)(a\tau)(a'\tau)(\alpha t)^2(\alpha' t)^2 \\ &\qquad = 2\{(b\tau_1)(b\tau_2)(b\tau)^2(\beta t)^4 - (\tau\tau_1)(\tau\tau_2).(\gamma t)^4\}, \\ 5^\circ &\equiv 9(aa')(\alpha\alpha')(a\tau_1)(a'\tau_1)(a\tau_2)(a'\tau_2)(\alpha t)^2(\alpha' t)^2 \\ &\qquad = 2\{(b\tau_1)^2(b\tau_2)^2(\beta t)^4 - (\tau_1\tau_2)^2.(\gamma t)^4\}, \\ 4^\circ &\equiv 3(aa')(\alpha\alpha')^3(a\tau_1)(a'\tau_2)(a\tau)(a'\tau) \\ &\qquad = 4\{(c\tau)^2(c\tau_1)(c\tau_2) - (\tau\tau_1)(\tau\tau_2).\delta\}, \\ 6^\circ &\equiv 3(aa')(\alpha\alpha')^3(a\tau_1)(a'\tau_1)(a\tau_2)(a'\tau_2) \\ &\qquad 4\{(c\tau_1)^2(c\tau_2)^2 - (\tau_1\tau_2)^2.\delta\}. \end{aligned}$$

The theorems which follow from these developments are contained in the next section.

9. The Forms F, $\bar{F}$ on a Conic. Counter Sextics.

We here interpret the form $F = (a\tau)^3(\alpha t)^3$ as a singly infinite number (for variable t) of triads of tangents τ of the given conic $K(\tau)$ and consider the locus $F(\tau)$ of vertices of these triangles. This locus $F(\tau)$ is of order six since a tangent τ is in three triads t and in each such triad τ, τ_1, τ_2 the tangent τ lies on the two vertices τ, τ_1 and τ, τ_2. Also the sextic locus $F(\tau)$ is birationally equivalent to $F = 0$ and therefore is of genus 4. For a point τ_1, τ_2 of $F(\tau)$ is in a unique triangle t, and the opposite side τ determines with t a unique solution of $F = 0$. Conversely if t, τ is a solution of $F = 0$, t will determine two other τ's, τ_1, τ_2 which fix a unique point of $F(\tau)$. Similarly the form $\bar{F} = (\bar{a}\tau)^3(\bar{\alpha}t)^3$ determines on the conic $K(\tau)$ in the plane S_x a sextic locus $\bar{F}(\tau)$ birationally equivalent to $\bar{F} = 0$. If in another plane S_y we fix the conic $K(t)$, the τ-triads of F and $\bar{F}$ determine triads of tangents of $K(t)$ whose vertices describe the sextic loci $F(t)$, $\bar{F}(t)$ also birationally equivalent to $F = 0$, $\bar{F} = 0$ respectively.

If $x = \tau_1, \tau_2$ is a point of $F(\tau)$ the cubics in t, $(a\tau_1)^3(\alpha t)^3$and $(a\tau_2)^3(\alpha t)^3$ have a common root t; or for the space cubics C_1, C_2 the plane t of C_2 is on the bisecant λ_x of C_1. Similarly if x is on $\bar{F}(\tau)$ the point t of C_2 is on the axis l_x of C_1. This according to **5**(23) identifies $\bar{F}(\tau)$ with the cusp locus $\Delta_{a,b}$ of perspective cubics of $\bar{S}_2(t)$ and therefore $F(\tau)$ with the cusp locus of perspective cubics of the counter rational sextic $S_1(t)$. If a triangle is circumscribed to a conic the lines joining each vertex to the contact of the opposite side meet in a point whose parameters are the hessian pair of the three parameters of the sides. For a t-triangle of the curve $F(\tau)$ this hessian is $(aa')^2(a\tau)(a'\tau)(\alpha t)^3(\alpha' t)^3$ whence the point is on $\bar{S}_2(t)$ [cf. **4**(5)]. Hence

(1) *On the sextic $\bar{F}(\tau)$ the τ-triads are the cusps of perspective cubics of*

$\bar{S}_2(t)$ *and the t-triads are triangles whose sides envelop* $K(\tau)$. *The point on the three lines which join the vertices of a t-triangle to the contacts of the opposite sides is a point of the counter sextic* $S_2(t)$. *The projective peculiarity of the sextic* $\bar{F}(\tau)$, *birationally general but subject to four projective conditions, is that its t-triads are triangles which envelop a conic.*

In proof of the last statement we note that if τ is a parameter on the conic then for each t there is determined three τ's; and every tangent τ, meeting the sextic in six points is determined by three t's so that the sextic is determined as above by a form $\bar{F}$ on $K(\tau)$. But such a form determines all the configurations used.

Naturally a theorem analogous to (1) applies to the other sextic $F(\tau)$ on S_x, as well as to the sextics $\bar{F}(t)$, $F(t)$ on S_y. We see from **7**(16) that the pair $\bar{F}(\tau)$, $\bar{F}(t)$ as well as the pair $F(\tau)$, $F(t)$ are quintic Cremona transforms of each other but on these transforms the role of the t- and τ-triads with reference to the norm conics are reversed.

We have obtained in **5**(10 and **7**(19) forms of the ternary equation of $\bar{F}(\tau)$. If in these we replace x by τ_1, τ_2 we have a form of degree 6 in τ_1 and in τ_2 which is the effective eliminant, after separation of the factor $(\tau_1\ \tau_2)^3$, of the two binary cubics $(\bar{a}\tau_1)^3(\bar{a}t)^3$, $(\bar{a}\tau_2)^3(\bar{a}t)$.[3]

The conic $K(\tau)$ has 12 points and 36 tangents in common with $\bar{F}(\tau)$. The common points arise when in $x=\tau_1$, τ_2 the two coincide. This happens for 12 values t, the branch points of the function $\tau(t)$ defined by $\bar{F}=0$. The branch values $\tau_1=\tau_2$ give rise to the 12 common points of $K(\tau)$ and $\bar{F}(\tau)$. Since the hessian of a triad with a double pair is the double pair these common points are on the counter sextic $S_2(t)$ also. Moreover the values τ residual to each branch value furnish a common tangent of $K(\tau)$ and $\bar{F}(\tau)$ since the tangent τ touches $\bar{F}(\tau)$ at the point $\tau, \tau_1=\tau, \tau_2$. Any tangent τ meets $\bar{F}(\tau)$ in three pairs of points each pair lying in a t-triad. This tangent touches $\bar{F}(\tau)$ when either the points of a pair come together which is the case considered above or when two pairs coincide. The second case occurs when for given τ two t's coincide. Then the line τ is a double tangent of $\bar{F}(\tau)$. These values τ are the branch points of the inverse function $t(\tau)$. The vertex opposite the side τ of such a t-triangle is one of the 12 points of theorem **5**(19). Hence

(2) *The sextic* $\bar{F}(\tau)$ *and the counter rational sextic* $S_2(t)$ *meet* $K(\tau)$ *in the same* 12 *points whose parameters on* $S_2(t)$ *are the branch points of the function* $\tau(t)$ *defined by* $\bar{F}=0$ *and on* $K(\tau)$ *are the branch values, and which on* $\bar{F}(\tau)$ *furnish the residual function value. The tangents to* $K(\tau)$ *at the* 12

points with the residual parameters are tangent also to $\bar{F}(\tau)$ at the coincidences of the linear series $g_1^3(t)$ upon it. The remaining 12 *common tangents of $K(\tau)$ and $\bar{F}(\tau)$ are double tangents of $\bar{F}(\tau)$ whose parameters on $K(\tau)$ are the branch points of the inverse function $t(\tau)$. The two further tangents to $K(\tau)$ from the contacts of such a double tangent meet in a point which is a coincidence of the linear series $g_1^3(\tau)$ on $\bar{F}(\tau)$, one of the* 12 *points described in* **5**(19).

We observe that the 12 coincidences of the two g_1^3's of $\bar{F}(\tau)$ are such that the 12 points of the one set are on a conic $c_{1,0}$, and that the tangents to $\bar{F}(\tau)$ at the 12 points of the other set envelop a conic $K(\tau)$.

We have found in **8**(5), (6), (9), (10), that the four forms $K \pm L \pm M \pm N$ $(\pm\pm\pm = +)$ for given τ_1, τ_2 are involution forms in t_1, t_2 i. e. that, given t_1, the two values of t_2 determined are such that with t_1 they form a triad and any pair of this triad will satisfy the symmetric (2, 2) form in t_1, t_2. Also we have found in 8(5), (6), (9′), (10′) that the same four forms for given t_1, t_2 are involution forms in τ_1, τ_2. The properties of such involution forms have been investigated by the writer in a series of papers on Symmetric Binary Forms and Involutions.* It there appears that a symmetric (2, 2) form, $(h_1t_1)^2(h_2t_2)^2 = (h_1t_2)^2(h_2t_1)^2$ is an involution form if the invariant

$$(3) \qquad \tfrac{1}{2}(h_1h'_1)(h_2h'_2)\{(h_1h_2)(h'_1h'_2)\} + \{(h_1h'_2)(h'_1h_2)\}$$

vanishes. Furthermore such a symmetric form defines (with reference to a norm conic $K(t)$) two conics—a *parametric conic* and an *apolarity conic.* The parametric conic is the locus of points $y = t_1, t_2$ for which t_1, t_2 satisfy $(h_1t_1)^2(h_2t_2)^2 = 0$. The apolarity conic is the locus of points $y = t_1, t_2$ for which t_1, t_2 are the roots of a quadratic $(kt)^2 = 0$ doubly apolar to the symmetric form i. e. $(h_1k)^2(h_2k')^2 = 0$. Thus the apolarity conic would be the parametric conic attached to the new symmetric form $(h_1t_1)(h_1t_2)(h_2t_1)(h_2t_2)$. When the symmetric form is an involution form determined by an I_1^3 on $K(t)$, the parametric conic is a locus of vertices of triangles circumscribed to $K(t)$ while the apolarity conic has for self polar triangles the points of the triads of I_1^3 on $K(t)$. The behavior of the two conics with respect to degenerate involutions is quite different. If I_1^3 has a neutral point i. e. if all its triads have a common member, the parametric conic breaks up into the tangent of $K(t)$ at the neutral point and another line; if I_1^3 has two neutral points the para-

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* This *Journal* (I), Vol. 31 (1909), p. 183; (II), Vol. 31 (1909), p. 355; (III), Vol. 32 (1910), p. 333. We are particularly concerned here with the content of (I) 2, 3, 5 (24), and 6. These papers will be cited hereafter as Sym. Forms I, II, III respectively.

metric conic breaks up into the two tangents of $K(t)$ at the two neutral points. On the other hand for one neutral point of I_1^3, the apolarity conic breaks up into two lines through the neutral point on $K(t)$; and for two neutral points the apolarity conic is the square of the line joining the two neutral points on $K(t)$. Thus the apolarity conic has the advantage that the cases of 0, 1, or 2 neutral points of I_1^3 are distinguished by the form of the conic.

The expanded form of the invariant (3) of the symmetric form

$(h_1t_1)^2(h_2t_2)^2 = t_1^2(a_{00}t_2^2 + 2a_{01}t_2 + a_{02}) + 2t_1(a_{10}t_2^2 + 2a_{11}t_2 + \cdots) + \cdots$
is

$$(4) \qquad a^2_{02} - a_{00}a_{22} + 4a_{01}a_{12} - 4a_{11}a_{02}.$$

This is the coincidence form of the bilinear invariant

$$(5) \quad \tfrac{1}{2}[2a_{02}a'_{02} - a_{00}a'_{22} - a_{22}a'_{00} + 4a_{01}a'_{12} + 4a_{12}a'_{01} - 4a_{11}a'_{02} - 4a_{02}a'_{11}].$$

This bilinear invariant when formed for two polarized quartics $(\beta t_1)^2(\beta t_2)^2$, $(\gamma t_1)^2(\gamma t_2)^2$ is $-\frac{1}{2}(\beta\gamma)^4$; for a polarized quartic and $(t_1t_2)^2$, vanishes; and for $(t_1t_2)^2$ with itself is 3. We now form the invariant (3) or (4) for the form $K+L+M+N$ which for given τ_1, τ_2 is a symmetric involution form in t_1, t_2; i. e. we form the bilinear invariant (5) of $K+L+M+N$ with itself. On making use of the facts just stated this invariant turns out to be

$$(6) \quad \begin{aligned} &-\tfrac{1}{2}\{(b\tau_1)^2(b\tau_2)^2(b'\tau_1)^2(b'\tau_2)^2(\beta\beta')^4 - 6[(c\tau_1)^2(c\tau_2)^2]^2\} \\ &-(\tau_1\tau_2)^2\{(b\tau_1)^2(b\tau_2)^2(\beta\gamma)^4 - 6\delta.(c\tau_1)^2(c\tau_2)^2\} \\ &-\tfrac{1}{2}(\tau_1\tau_2)^4\{(\gamma\gamma')^4 - 6\delta^2\}. \end{aligned}$$

Since this is to vanish for all values of τ_1, τ_2 we expand according to the Clebsch-Gordan theorem and equate to zero the coefficients of 1, $(\tau_1\tau_2)^2$, $(\tau_1\tau_2)^4$ getting

$$(7) \quad \begin{aligned} &-\tfrac{1}{2}\{(b\tau)^4(b'\tau)^4(\beta\beta')^4 - 6[(c\tau)^4]^2\} = 0, \\ &\tfrac{1}{7}\{(bb')^2(b\tau)^2(b'\tau)^2(\beta\beta')^4 - 6(cc')^2(c\tau)^2(c'\tau)^2\} \\ &\qquad\qquad - \{(b\tau)^4(\beta\gamma)^4 - 6\delta(c\tau)^4\} = 0, \\ &-\tfrac{1}{60}\{(bb')^4(\beta\beta')^4 - 6(cc')^4\} - \tfrac{1}{2}\{\gamma\gamma')^4 - 6\delta^2\} = 0. \end{aligned}$$

Referring to the theorem **3**(18) we see that the equations (7) are actually satisfied due to the syzygies of the second degree satisfied by the coefficients of the four forms K, L, M, N. But we see also that the conditions (7) together with the corresponding ones obtained from $K-L-M+N$ which arise from (7) by the change of sign of $(c\tau)^4$ and $(\gamma t)^4$; together also with the corresponding two conditions (6) that $K \pm L \pm M + N$ are involution forms in τ_1, τ_2 for given t_1, t_2 entail the existence of all but one of the syzygies **3**(18) whence

(8) *The four conditions that the two forms $K+L+M+N$ and $K-L-M+N$ are, for given τ_1, τ_2 or for given t_1, t_2, involution forms in the remaining two variables, are sufficient to identify the coefficients of these forms as the two-row minors of a four-row determinant.*

We now consider as a sample of the above four involutions the particular one $\overline{\Gamma}$, or $K-L-M+N$ which is determined by the two binary cubics, $(\bar{a}\tau_1)^3(\bar{a}t)^3$ and $(\bar{a}\tau_2)^3(\bar{a}t)^3$, which determine on the space cubic C_2 the triads of points cut out by the planes τ_1, τ_2 of C_1 i. e. it is the involution cut out on C_2 by the pencil of planes on the axis l_x $(x=\tau_1,\tau_2)$ of C_1. If the involution has one neutral point t the axis l_x is on a point t of C_2; if the involution has two neutral points t, t' the axis l_x of C_1 is a bisecant t, t' of C_2. Thus from **5**(23) the locus of x for the case of one neutral point is $\bar{F}(\tau)$. We have seen that the condition for one neutral point is the vanishing of the discriminant of the apolarity involution conic while the condition for two neutral points is the identical vanishing of the line equation of the apolarity conic. Hence we may state the theorem:

(9) *If in the form $K-L-M+N$ we replace τ_1, τ_2 by x; if we polarize t_1^2 into t_1, t_2 and t_2^2 into t_1, t_2; and replace t_1, t_2 by y; then we obtain a form*

$$(ex)^2(fy)^2$$

*which for given x is the apolarity involution conic on S_y whose two self-polar triangles on $K(t)$ are determined by the cubics $(\bar{a}\tau_1)^3(\bar{a}t)^3$ and $(\bar{a}\tau_2)^3(\bar{a}t)^3$. The equation on S_x of the sextic $\bar{F}(\tau)$ is $(ex)^2(e'x)^2(e''x)^2(ff'f'')^2=0$. For x a node of $\bar{F}(\tau)$ the form $(ex)^2(e'x)^2(ff'\eta)^2$ vanishes identically whence for variable η in $(ex)^2(e'x)^2(ff'\eta)^2=0$ we have a system of contact adjoint quartics of $\bar{F}(\tau)$ whose I_2^6 of contacts is cut out by the system of adjoint quartics determined by variable η' for fixed η in $(ex)^2(e'x)^2(ff'\eta')(ff'\eta)=0$. On the other hand the conic $(cx)^2(dy)^2=0$ of **7**(10) is for given x the parametric involution conic on two τ-triads of $\bar{F}(\tau)$.*

The equation of this form $(ex)^2(fy)^2$ before τ_1, τ_2 and t_1, t_2 are replaced by x, y respectively in the notation of **8**(5) and (7) is

$$\begin{aligned}(10)\quad (ex)^2(fy)^2 &= (b\tau_1)^2(b\tau_2)^2(\beta t_1)^2(\beta t_2)^2+\tfrac{1}{2}(t_1t_2)^2.(c\tau_1)^2(c\tau_2)^2\\ &\quad -(\tau_1\tau_2)^2.(\gamma t_1)^2(\gamma t_2)^2-\tfrac{1}{2}(t_1t_2)^2.(\tau_1\tau_2)^2.\delta=\tfrac{9}{4}(c_4+c_5)=\\ &\quad \tfrac{9}{4}(aa')(\alpha\alpha')(a\tau_1)(a'\tau_1)(a\tau_2)(a'\tau_2)(\alpha t_1)(\alpha' t_2)\\ &\qquad [(\alpha t_1)(\alpha' t_2)+(\alpha t_2)(\alpha' t_1)].\end{aligned}$$

If in theorem (9) we replace in $K-L-M+N$ the t_1, t_2 by y and proceed

similarly the sextic $\bar{F}(t)$ in S_y is obtained; whereas for $K+L+M+N$ we obtain sextics $F(\tau)$ in S_x and $F(t)$ in S_y.

When x is a point on $\bar{F}(\tau)$ for which $(\bar{a}\tau_1)^3(\bar{a}t)^3$ and $(\bar{a}\tau_2)^3(\bar{a}t)^3$ have a common root t then the involution conic $(ex)^2(fy)^2=0$ has a node at the point t of the norm-conic $K(t)$. The same node is obtained from any pair of τ's of the t-triad $(\bar{a}\tau)^3(\bar{a}t)^3=0$ on $\bar{F}(\tau)$. Hence we conclude that

(11) *As x runs over the sextic $\bar{F}(\tau)$ the point y in $(ex)^2(e'x)^2(ff'\eta)^2\equiv(y\eta)^2$ runs over the conic $K(\tau)$ three times each point t of the conic corresponding to a t-triad of $\bar{F}(\tau)$. The contacts of the system $(ex)^2(e'x)^2(ff'\eta)^2=0$ with $\bar{F}(\tau)$ occur at the two t-triads determined by the intersections of the line η with $K(t)$; and in the system there is a one-parameter quartic system of adjoints determined by the tangents η of $K(t)$ which have four-point contact at the points of a t-triad on $\bar{F}(\tau)$.*

According to Sym. Forms I (32) the apolarity involution conic in lines is apolar to the norm conic whence its line equation (l. c. (31)) is a symmetric 2, 2 form which is merely a polarized binary quartic. The equation of this quartic in t is

$$(12)\quad \begin{aligned}&(b\tau_1)^2(b\tau_2)^2(b'\tau_1)^2(b'\tau_2)^2(\beta\beta')^2(\beta t)^2(\beta' t)^2\\ &-2(\tau_1\tau_2)^2.(b\tau_1)^2(b\tau_2)^2(\beta\gamma)^2(\beta t)^2(\gamma t)^2\\ &-\tfrac{1}{2}[(c\tau_1)^2(c\tau_2)^2-(\tau_1\tau_2)^2.\delta][(b\tau_1)^2(b\tau_2)^2(\beta t)^4-(\tau_1\tau_2)^2.(\gamma t)^4]\\ &+(\tau_1\tau_2)^4.(\gamma\gamma')^2(\gamma t)^2(\gamma' t)^2=0.\end{aligned}$$

Thus for given t in (12) and $x=\tau_1,\tau_2$ we have the equation of the adjoint quartic of $\bar{F}(\tau)$ which has 4-point contact at the three points of the t-triad; if however (12) is polarized as to t_1, t_2, t_3, t_4 we have that adjoint quartic which cuts $\bar{F}(\tau)$ in the four corresponding t-triads.

If we apply the same reasoning to the involution **8**(9) we have corresponding developments for the rational sextic $\bar{S}_2(t)$. This involution is determined at x by the triads of tangents of perspective cubics of $\bar{S}_2(t)$ which pass through x and in connection with **5**(19) we have seen that the involution has one neutral point at a point of $\bar{S}_2(t)$ and two neutral points at a node of $\bar{S}_2(t)$. We may then state at once the theorem:

(13) *If, in the form $K+L-M-N$, we replace τ_1, τ_2 by x; if we polarize t_1^2 into t_1, t_2 and t_2^2 into t_1, t_2; and then replace t_1, t_2 by y, we obtain a form*

$$(14)\quad \begin{aligned}(gx)^2(hy)^2&=(b\tau_1)^2(b\tau_2)^2(\beta t_1)^2(\beta t_2)^2-\tfrac{1}{2}(t_1t_2)^2.(c\tau_1)^2(c\tau_2)^2\\ &\quad-(\tau_1\tau_2)^2.(\gamma t_1)^2(\gamma t_2)^2+\tfrac{1}{2}(t_1t_2)^2.(\tau_1\tau_2)^2.\delta\\ &=\tfrac{3}{4}(c_4+5c_5)\end{aligned}$$

which for given x is the apolarity involution conic on S_y whose self polar triangles on $K(t)$ are determined by the triads of tangents on x of perspective cubics of $\bar{S}_2(t)$. The equation on S_x of the sextic $\bar{S}_2(t)$ is $(gx)^2(g'x)^2(g''x)^2 (hh'h'')^2 = 0$. For x a node of $\bar{S}_2(t)$ the form $(gx)^2(g'x)^2(hh'\eta)^2$ vanishes identically and the form $(gx)^2(g'x)^2(hh'c)^2 = 0$ ($(cy)^2$ an arbitrary conic) for variable c furnishes the system of ∞^4 adjoint quartic curves of the rational sextic $\bar{S}_2(t)$. For given x on $\bar{S}_2(t)$ the form $(gx)^2(g'x)^2(hh'\eta)^2 \equiv (y\eta)^2$ maps $\bar{S}_2(t)$ upon the conic $K(t)$ in S_y.

We note that the system of quartic curves is only four-fold since the line form of an apolarity involution conic is necessarily apolar to the norm conic $K(t)$ so that $(gx)^2(g'x)^2(hh'K)^2$ is identically zero in x.

Again writing the line equation of the apolarity conic as a quartic in t we have

(15) *The form*

$$\begin{aligned}&(b\tau_1)^2(b\tau_2)^2(b'\tau_1)^2(b'\tau_2)^2(\beta\beta')^2(\beta t)^2(\beta' t)^2\\ &\qquad\qquad - 2(\tau_1\tau_2)^2.(b\tau_1)^2(b\tau_2)^2(\beta\gamma)^2(\beta t)^2(\gamma t)^2\\ &+ \tfrac{1}{2}[(c\tau_1)^2(c\tau_2)^2 - (\tau_1\tau_2)^2.\delta][(b\tau_1)^2(b\tau_2)^2(\beta t)^4 - (\tau_1\tau_2)^2.(\gamma t)^4]\\ &+ (\tau_1\tau_2)^4.(\gamma\gamma')^2(\gamma t)^2(\gamma' t)^2 = 0\end{aligned}$$

for given t and $x = \tau_1, \tau_2$ is the adjoint quartic curve of $\bar{S}_2(t)$ which has four-fold contact at the point t; polarized for t as to t_1, t_2, t_3, t_4 it is the adjoint quartic curve on the ten nodes and four points $t_1, \cdots t_4$ of $\bar{S}_2(t)$.

We shall find in **12** another use for this binary quartic in t in connection with the perspective rational quartic envelopes of $\bar{S}_2(t)$.

We remark that theorems entirely analogous to (13) and (15) hold for $S_2(t)$ when we begin with the form $K - L + M - N$; for $S_1(\tau)$, when we begin with the form $K - L + M - N$ and replace t_1, t_2 by y; and for $\bar{S}_1(\tau)$ when we begin with the form $K + L - M - N$ and replace t_1, t_2 by y as is evident from the comparison of **8**(10), (9′), (10′) with **8**(9).

Included in the list in **5** of comitants of the second degree of $(\pi x)(a\tau)(at)^3$ are two conics namely 4° and 6°. The conic 4° is the locus of the point of intersection of the three cusp tangents of a perspective cubic; the conic 6° is $c_{1,0}$ of **5**(19) on which the sextics $\bar{S}_2(t)$ and $\bar{F}(\tau)$ osculate. Because of this relation we shall denote the conic 6° by $\bar{c}$ and the conic 4° by $\bar{d}$ and denote the corresponding conics of the counter rational sextic $S_2(t)$ by c and d respectively. On making use of formulae **8**(11), (12) we find for the equations of these conics

$$\bar{c} = (aa')(aa')^3(a\tau_1)(a\tau_2)(a'\tau_1)(a'\tau_2) = \tfrac{4}{3}\{(c\tau_1)^2(c\tau_2)^2 - (\tau_1\tau_2)^2.\delta\}$$
$$\bar{d} = (aa')(aa')^3(a\tau')(a'\tau')(a\tau)(a'\tau) = \tfrac{4}{3}\{(c\tau)^2(c\tau')^2 - (\tau\tau')^2.\delta\}$$
$$(x = \tau_1, \tau_2),$$

where in $\bar{d}$ for given τ the point x of $\bar{d}$ is determined by the roots τ_1', τ_2' of the quadratic in τ'. Since $S_2(t)$ arises from $\bar{S}_2(t)$ by a change of sign of $(c\tau)^4$, $(\gamma t)^4$ the corresponding conics c, d of $S_2(t)$ have equations

$$c = -\tfrac{4}{3}\{(c\tau_1)^2(c\tau_2)^2 + (\tau_1\tau_2)^2.\delta\} \quad (x = \tau_1, \tau_2),$$
$$(17) \qquad d = -\tfrac{4}{3}\{(c\tau)^2(c\tau')^2 + (\tau\tau')^2.\delta\}$$
$$(x \text{ a quadratic in } \tau' \text{ for given } \tau).$$

We prove first that these four conics are involution conics with respect to the norm-conic $K(\tau)$. In fact a general symmetric (2, 2) form $(h_1t_1)^2(h_2t_2)^2$ may be expressed as the sum of a polarized quartic and a term in $(t_1t_2)^2$ i. e. as $(ht_1)^2(ht_2)^2 + k(t_1t_2)^2$ and this is an involution conic if $(hh')^4 = 6k^2$. For the symmetric forms in (16), (17) this condition becomes $(cc')^4 = 6\delta^2$ which according to **3**(18) is satisfied for all four conics. But the interpretation of the parameters is different in the cases of conics $\bar{c}$, c and conics $\bar{d}$, d. Thus we see that $\bar{c}$ is a parametric involution conic whereas $\bar{d}$ is an apolarity involution conic. Hence we have the theorem:

(18) *The four conics, c, d. $\bar{c}$, $\bar{d}$ lie in a pencil with $K(\tau)$ and meet $K(\tau)$ in the tetrad of points $(c\tau)^4 = 0$. They are the four apolarity and parametric involution conics with respect to the norm-conic $K(\tau)$ which pass through this tetrad. Thus $\bar{d}$, the locus of the point of intersection of the three cusp tangents of perspective cubics of $\bar{S}_2(t)$, contains the vertices of triangles circumscribed to $K(\tau)$, and the contacts of these triangles with $K(\tau)$ are self polar triangles of $\bar{c}$.*

In this section then we have obtained the equations of the four sextics F of genus 4 and the four rational sextics S, as of degree three in the four self dual comitants. They here appear in remarkably symmetric fashion. It is to be noted however that this symmetry is due in part to the use of the Darboux system of coördinates with reference to the norm-conics $K(\tau)$, $K(t)$.

As preliminary to the study of the *conjugate* rational sextic curve in space whose plane sections are apolar to the line sections of the plane rational sextic we introduce in the next two sections, the jacobian quartic surface of a web of quadrics, the symmetroid quartic surface or symmetric four-row determinant of linear forms, and the so-called Stahl quadric, surfaces which are covariantly related to the rational sextic curve.

10. The Jacobian and Symmetroid Quartic Surfaces. The Form $\binom{1\,2}{3\,3}$.

In order to avoid confusion we shall in this section use x, ξ as point and line coördinates in the plane of the sextic $\bar{S}_2(t)$; y, η as point and plane coördinates in the space S_3 of the cubic curves C_1, C_2; and z, ζ as point and plane coördinates in the space S_3 of the symmetroid Σ.

We have already noted in **4** that the nets $\bar{Q}_1$, $\bar{Q}_2$ of quadric envelopes on the curves $\bar{C}_1$, $\bar{C}_2$ respectively have no common quadrics and that there is therefore a web of point quadrics

$$(1) \qquad \zeta_0 Q_0(y) + \zeta_1 Q_1(y) + \zeta_2 Q_2(y) + \zeta_3 Q_3(y) = 0$$

which is apolar to the system $\bar{Q}_1 + \bar{Q}_2$. It is well known that

(2) *The jacobian J of the four quadrics Q_i is the locus of nodes of quadrics of the web* (1) *or the locus of pairs of points y, y' apolar to the web. These pairs lie in an involutory correspondence I on J. The surface J is a quartic surface on the ten common axes of C_1, C_2.*

The first three of these properties follow from the usual theory of jacobians and the last partly from the fact that a web of quadrics contains ten pairs of planes whose double lines we shall presently identify with the common axes.

Let the quadric $\bar{Q}_1$ on C_1 meet C_2 in planes $t_1, \cdots, t_6$, a hexad of the involution $\bar{I}_2^6$ cut out on $\bar{S}_2(t)$ by the lines of the plane. Then the entire web $Q(y)$ will meet C_2 in sets of 6 points in the conjugate $\bar{I}_3^6$. For $Q(y)$ is apolar to $\bar{Q}_2$ and therefore the polarized quadrics of $Q(y)$ will be represented on C_2 by polarized sextics; and $Q(y)$ is apolar to $\bar{Q}_1$ and therefore these sextics are apolar to the sextics cut out on C_2 by $\bar{Q}_1$ i. e. to $\bar{I}_2^6$. Now the linear triad t_1, t_2, t_3 of $\bar{S}_2(t)$ can be supplemented by a triad t_4, t_5, t_6 to form a hexad apolar to $\bar{I}_3^6$ whence the polars of the triad t_1, t_2, t_3 as to $\bar{I}_3^6$ will be linearly related or the triad will be apolar to a unique hexad of $\bar{I}_3^6$ cut out say by $Q'(y)$. Then $Q'(y)$ can be represented as a sum of squares of the three planes t_1, t_2, t_3 of C_2 and has a node at the point $y(t_1, t_2, t_3)$ on these three planes. Conversely if $Q'(y)$ has a node y it cuts C_2 in a hexad whose catalecticant (the discriminant of $Q'(y)$) vanishes whence $Q'(y)$ can be expressed as a sum of three squares of planes t. Moreover the quadric of the net $\bar{Q}_1$ which touches two of these planes must touch the third since $\bar{Q}_1$ is apolar to $Q'(y)$ whence the planes t_1, t_2, t_3 are a triad of a hexad of $\bar{I}_2^6$. Thus the surface J is the locus of points y on planes t_1, t_2, t_3 where t_1, t_2, t_3 satisfy the symmetric condition

(3) $$\bar{f}_2(t_1^4, t_2^4, t_3^4) = 0$$

which expresses that t_1, t_2, t_3 are a linear triad on the plane rational sextic $\bar{S}_2(t)$. Similarly y is on the three planes τ_1, τ_2, τ_3 of C_1 such that τ_1, τ_2, τ_3 are a linear triad on the paired rational sextic $\bar{S}_1(\tau)$ and satisfy the collinear condition

(4) $$\bar{f}_1(\tau_1^4, \tau_2^4, \tau_3^4) = 0.$$

Moreover the two trihedrals t_1, t_2, t_3 and τ_1, τ_2, τ_3 on y, being self polar as to $Q'(y)$ are such that the six planes touch a quadric cone $q'(y)$ with vertex at y. Conversely if y is a point for which the two trihedrals have this property then, of the ∞^2 quadrics $\bar{Q}_1$ on τ_1, τ_2, τ_3 one is on t_1, t_2 and therefore contains all of the planes of $q'(y)$ including t_3. Hence

(5) *The jacobian quartic surface J is the locus of points y for which the two sets of three planes of C_1, C_2 on y are planes of a quadric cone. The two trihedrals on C_1, C_2 correspond to linear triads on the paired rational sextics $\bar{S}_1(\tau)$, $\bar{S}_2(t)$. To every such linear triad there corresponds one and only one point of J which is the parametric involution surface both of $\bar{S}_2(t)$ when referred to C_2 and of $\bar{S}_1(\tau)$ when referred to C_1, the involution forms being* (3) *and* (4). *If the point y on J is determined by the cubic $(c\tau)^3$ on C_1 and $(\gamma t)^3$ on C_2 then $(ca)^3(at)^3 = (\gamma t)^3$ and $(\bar{a}\tau)^3(\gamma\bar{a})^3/R = (c\tau)^3$.*

The last statement follows from **4**(6). We shall identify later in **11** the locus on J for which the planes of the quadric cone degenerate into two plane pencils. Since, for t_1, t_2 nodal parameters of $\bar{S}_2(t)$ and t any other point, the ∞^1 triads t_1, t_2, t are collinear the corresponding points of J are on the axis t_1, t_2 of C_2 which is an axis of C_1 also. This identifies the ten lines on J.

We have seen that there is one quadric $\bar{q}_1$ of the net $\bar{Q}_1$ on the planes τ_1, τ_2, τ_2, t_1, t_2, t_3 which contains the planes of the cone $q'(y)$. Similarly there is a quadric $\bar{q}_2$ of the net $\bar{Q}_2$ on the same six planes and cone. Each of the quadrics $\bar{q}_1$, $\bar{q}_2$ also contains the planes τ_4, τ_5, τ_6, t_4, t_5, t_6 and therefore the cone $q'(y')$ on the second set of 6 planes. But if two quadric envelopes have two quadric cones in common a member of their pencil factors into the product y, y' of the vertices of the cones. Hence y, y' are corresponding points of I on J and the corresponding linear triads on either rational sextic make up a linear hexad on that sextic. Thus if, on $\bar{S}_2(t)$, t_1, t_2 are nodal parameters on a line with t_3, t_4, t_5, t_6 then as the line revolves about the node the tetrads t_3, t_4, t_5, t_6 determine on C_2 an I_1^4 whose triads are marked by a cubic curve on J. If t_3, t_4 are the parameters of another node the cubic curve of I_1^4 meets the axis t_3, t_4 of C_2 in the points t_3, t_4, t_5 and t_3, t_4, t_6 or this axis is a bisecant of the involution cubic curve. Hence

(6) *The ten common axes of C_1, C_2 on J have as correspondents under I ten cubic curves on J each of which has as bisecants the nine axes other than the corresponding one. These cubic curves referred either to C_1 or C_2 are involution curves. A co-pair of I on J arises from complementary linear triads on the same line section of either sextic $\bar{S}_2(t)$, $\bar{S}_1(\tau)$.*

Taking account of the 20 linear triads that can be formed from a line section of $\bar{S}_2(t)$ or $\bar{S}_1(\tau)$ we see that

(7) *There are two systems of ∞^2 6-planes, one on each of the curves C_1, C_2, inscribed in J. Each point y of J determines one 6-plane of either system and these two 6-planes have in common also the opposite point y' of the 6-planes, the co-point of y under I.*

We shall find in connection with the symmetroid two such systems of inscribed 6-planes which envelop the Stahl quadric rather than a cubic curve.

In discussing the birational transformation from J to Σ it is more convenient to regard Σ as a quartic locus of points though in connection with a rational point sextic curve in space Σ turns up as a quartic envelope of planes.

For the moment then we replace the coördinates ζ in (1) by point coördinates z and thereby obtain a general $\binom{1\,2}{3\,3}$ form in quaternary coördinates z, y

(8) $$(\alpha y)^2(\beta z).$$

Such a form has 40 coefficients and depends upon 39 constants or $39 - 15 - 15 = 9$ absolute projective constants. For every point z in S_z the form (8) determines a quadric in S_y and the locus of points z for which this quadric has a node y is the symmetroid

(9) $$\Sigma = (\beta z)(\beta' z)(\beta'' z)(\beta''' z)(\alpha\alpha'\alpha''\alpha''')^2 = 0,$$

a symmetric four-row determinant whose elements are linear forms in z. Evidently any such determinant may be regarded as the discriminant of a quadric (8). For point z on Σ and node y of quadric (8) the equations

(10) $$\alpha_i(\alpha y)(\beta z) = 0 \quad (\mathrm{i} = 0, 1, 2, 3)$$

are simultaneously satisfied. On eliminating z we have

(11) $$J = (\beta\beta'\beta''\beta''')(\alpha\alpha'\alpha''\alpha''')(\alpha y)(\alpha' y)(\alpha'' y)(\alpha''' y) = 0,$$

the jacobian J of the web (8) for variable z. For corresponding points y, y' of J under I the equations

(12) $$\beta_i(\alpha y)(\alpha y') = 0 \quad (i = 0, 1, 2, 3)$$

hold simultaneously. For z, y corresponding points on Σ, J we have for the point y the equation

$$(13) \qquad (\beta z)(\beta' z)(\beta'' z)(aa'a''\eta)^2 = (y\eta)^2.$$

Any point y in (8) determines a plane ζ, and a plane ζ on points z_1, z_2, z_3 is determined by any one of the 8 base points y of the net of quadrics $(ay)^2(\beta z_i)=0$ $(i=1,2,3)$ on y. Thus (8) defines an $(8-1)$ correspondence between points y and planes ζ of the spaces of J, Σ. If y is a point of J one of the three quadrics of the web on y has a node at y and two of the 8 points corresponding to ζ coincide at y. If z_0 is the point of Σ which furnishes this nodal quadric then the square of y is furnished by (13) and this substituted in (8) yields the plane ζ which corresponds to y in the form $(\beta z_0)(\beta' z_0)(\beta'' z_0)(\beta''' z)(aa'a''a''')^2=0$ which is the tangent plane of Σ at the point z_0 which corresponds to y. Hence the form (8) maps J upon the planes ζ of the symmetroid Σ. As an envelope Σ is of class 16 since if ζ in (8) is on z, z' then y is on a quartic space curve which meets J in 16 points.

Since there is in the web (8) 10 quadrics which are pairs of planes whose nodes are the ten lines of J there are 10 points z_0 for which (13) vanishes identically and at which therefore the tangent plane of Σ is indeterminate. Hence

(14) *The form* (8) *maps the points of the surface J upon the planes of the symmetroid Σ of class* 16 *and order* 4 *in such a way that point y and contact z of the tangent plane ζ correspond in* (10). *The ten lines y of J map into the planes of the ten tangent quadric cones of Σ at its ten nodes or the lines of J correspond to the directions on Σ at its ten nodes.*

A quadric $(\delta y)^2$ meets J in a curve which according to (13) corresponds to a curve on Σ which is cut out by the surface $(\beta z)(\beta' z)(\beta'' z)(aa'a''\delta)^2=0$. This cubic equation is evidently satisfied by the nodes whence taking account of the particular cases where $(\delta y)^2$ degenerates into a pair of planes or a repeated plane we have

(15) *Under the birational transformation B in* (10) *between J and Σ a quadric section by $(\delta y)^2$ corresponds to the section of Σ by its cubic adjoint surface, $(\beta z)(\beta' z)(\beta'' z)(aa'a''\delta)^2=0$. The sections of J by planes η correspond on Σ to a linear system of curves Γ_3^6 of order* 6 *and genus* 3 *on the nodes of Σ which are the contact curves of the cubic adjoints $(\beta z)(\beta' z)(\beta'' z)$ $(aa'a''\eta)^2=0$, whence on every plane section π of Σ there is isolated a system*

of contact cubics. The linear system Γ_3^6 *is cut out on* Σ *by the linear system of adjoint cubics* $(\beta z)(\beta' z)(\beta'' z)(aa'a''\eta)(aa'a''\eta')=0$ *for variable* η *and fixed* η'.

If ζ is a plane section of Σ determined by points z, z', z'' then the three quadrics $(ay)^2(\beta z^{(i)})$ determine a net and the involutory Cremona transformation determined by this net is

$$(aa'a''\eta)(\beta\beta'\beta''\zeta)(ay)(a'y)(a''y)=0. \tag{16}$$

The singular sextic curve of this involution is the member of the linear system C_3^6 on J which corresponds under B to the plane section ζ of Σ. On the surface J this is the involution I. It transforms the plane section η into the intersection of the cubic surface (16) with J which is residual to C_3^6. Hence

(17) *The plane sections of* Σ *correspond under* B *to a linear system* C_3^6 *of sextics of genus three on* J*, the loci of nodes of nets of quadrics in the web* (8). *Cubic surfaces on* C_3^6 *cut* J *in a residual system* C'^6_3 *of sextics of genus* 3, *the transforms of plane sections of* J *under* I. *For given sections* ζ *of* Σ *and* η *of* J *the cubic surface* (16) *cuts* J *in the curves* C_3^6, C'^6_3 *which correspond to* ζ, η *under* B, I *respectively.*

If then in (16) we fix η and let ζ vary the curve C_3^6 which corresponds to ζ under B is cut out by cubic surfaces on the fixed C'^6_3 determined as the transform of the section η by I. Hence

(18) *If* ζ *corresponds to* C_3^6 *on* J *under* B *and* η *corresponds to* Γ_3^6 *on* Σ *under* B *but to* C'^6_3 *under* I *then for given* ζ (16) *is a Cremona involution which effects on* J *the involution* I *and has for singular curve* C_3^6*; whereas for given* η (16) *is a Cremona transformation of* J *into* Σ *which has* C'^6_3 *and* Γ_3^6 *for singular curves. The inverse transformation is obtained from* (13) *in the form* $(\beta z)(\beta' z)(\beta'' z)(aa'a''\eta)(aa'a''\eta')=0$ *for fixed* η'. *Thus on any plane section of* J *there is isolated two coresidual linear series* g_3^6 *cut out respectively by the linear systems* C_3^6 *and* C'^6_3.

To the points y on the cubic curves C_1, C_2 there correspond in (8) the planes ζ of two space rational sextic curves $\overline{R}_1(\tau)$, $\overline{R}_2(t)$. The planes ζ of these curves on a point z are given by the parameters of the point in which the quadric (8) cuts C_1, C_2 whence the sextic curves $\overline{R}_1(\tau)$, $\overline{R}_2(t)$ are the conjugate rational curves of $\bar{S}_1(\tau)$, $\bar{S}_2(t)$ respectively. The symmetroid Σ is the locus of points z for which the quadric (8) can be expressed as a sum of squares of three planes of C_1 or three planes of C_2 and therefore is the locus

of points z whose point sections of $\overline{R}_1(\tau)$ or $\overline{R}_2(t)$ are expressible as a sum of three sixth powers or are *catalectic sextics.* If however z is a node of Σ the quadric (8) is a pair of planes η, η' on a common axis of C_1, C_2. Since η, η' is apolar to the web $\bar{Q}_1$ the planes η, η' are harmonic to the pairs of planes of C_1 and C_2 on their common axis and the quadric can be expressed as a sum of squares of either pair of planes. Hence the sextic point-section of $\overline{R}_1(\tau)$ or of $\overline{R}_2(t)$ from a node of Σ is a *cyclic sextic,* i. e. a sextic reducible to a sum of two sixth powers of linear forms. The linear forms themselves determine a pair of nodal parameters on the conjugate plane sextics $\bar{S}_1(\tau)$, $\bar{S}_2(t)$.

It is hoped that the remaining sections of Part I will appear in an early number.

THE UNIVERSITY OF ILLINOIS,
Oct. 1, 1923.

The Curve of Ambience.

By Frank Morley.

The Curve of Pursuit for a line is well known, see for instance Professor Cohen's "Differential Equations." That for a circle has received some attention as leading to a differential equation which is not integrable in the ordinary sense. References are given in the *American Mathematical Monthly*, Vol. 28, Feb. 1921, and in the same number are two studies of graphical integration for this case by Professor A. S. Hathaway and by my son F. V. Morley. The problem is that of a dog running after a man, velocities being constant. It occurred to me that the case of a dog D running around a man M might be more tractable. What is meant is that the path of D is always at right angles to the line DM. For a given locus of M any one of the two-parameter system of curves for D may be called a *curve of ambience.*

§ 1. *The general equations.*

Let two directed curves make angles θ and θ_1 with a ray or directed line whose angle (that is the angle which it makes with a fixed ray) is ϕ. Let s and s_1 be the lengths of the curves, n the length of the intercept on the ray. Then the variations of n, ϕ are given by

1.1) $$dn = ds_1 \cos\theta_1 - ds\cos\theta,$$

1.2) $$nd\phi = ds_1 \sin\theta_1 - ds \sin\theta.$$

The equations are written down by resolving the velocities of the end-points, but a rigid proof if desired is obtained by considering the variations for a triangle. If the sides be r_1, r_2, r_3 and the angles of the sides ϕ_1, ϕ_2, ϕ_3, so that

$$\Sigma\, re^{\iota\phi} = 0,$$

then $$\Sigma\, re^{\iota\phi}(dr/r + \iota d\phi) = 0.$$

In our case, where MD is the interecept, always normal to the path of D, $\theta_1 = \pi/2$; and further ds_1/ds is a constant κ, so that

1.3) $$dn = - ds\cos\theta,$$

1.4) $$nd\phi = ds(\kappa - \sin\theta).$$

These are the differential equations of the general problem.

We shall suppose that the man is always on the left of the dog. First we consider the cases where M describes a line, using rectangular coördinates X, Y. Let M describe the vertical axis, $X = 0$. Then X being the abscissa of D,

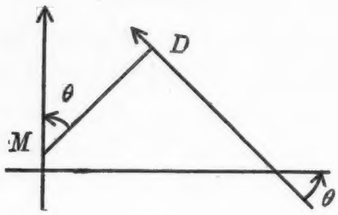


FIG. 1.

$$\frac{dX}{ds_1} = -\cos\theta$$

or

$$\frac{dX}{\kappa ds} = -\cos\theta$$

But the equation 1.3 is

$$dn = -ds\cos\theta:$$

Hence $$dX = \kappa dn,$$

1.5) $$\kappa n = X - X_0.$$

In words the distance DM is proportional to the distance to D from the vertical line $X = X_0$; thus the relative curve is a conic with focus M, eccentricity $1/\kappa$, and focal axis horizontal.

§ 2. *The parabolic case.*

To pass from the relative path to the actual, we employ a general method, in place of using the second differential equation. Take first the parabolic case, $\kappa = 1$. The line of reals ρ is mapped on a parabola by the equation

$$x = (\rho + \iota)^2$$

To displace this vertically we add a term ιs, where the real s is the distance gone by the man. Thus

2.1) $$x = (\rho + \iota)^2 + \iota s,$$
$$dx = 2(\rho + \iota)\,d\rho + \iota ds.$$

We select the point x which is moving at right angles to the focal ray by writing $dx \perp x - \iota s$, so that

2.2) $$\frac{2(\rho + \iota)\,d\rho + \iota ds}{2(\rho - \iota)\,d\rho - \iota ds} = -\left(\frac{\rho - \iota}{\rho + \iota}\right)^2,$$

whence $$ds = -(\rho^2 + 1)\,d\rho$$

or with $s = 0$ when $\rho = 0$, $s = -\rho^3/3 - \rho$

$$x = \tfrac{1}{3}(\iota\rho)^3 - \iota\rho + (\rho + \iota)^2,$$

2.3) $$3x + 2 = (\iota\rho - 1)^3.$$

This then is the curve of ambience for a line, in the case of equal velocities. It is projectively speaking a rational cubic. Other forms of the equation are, if $-y = 3x + 2$,

$$y^{1/3} + \bar{y}^{1/3} = 2\ ,$$

or in polar coördinates

$$r^{1/3} \cos\theta/3 = 1\ ,$$

or as a map of the base circle $|\,t\,| = 1$

$$y = \frac{8}{(1 + t)^3}\ .$$

This last form would have been obtained had we written the parabola as the map of the base circle,

$$x = \frac{1}{(1 + t)^2}\ .$$

The rectangular coördinates of the dog are, from 2.3),

2d) $$X = \rho^2 - 1$$
$$3Y = 3\rho - \rho^3\ ,$$

those of the man being

$$X_1 = 0$$
2m) $$3Y_1 = -3\rho - \rho^3.$$

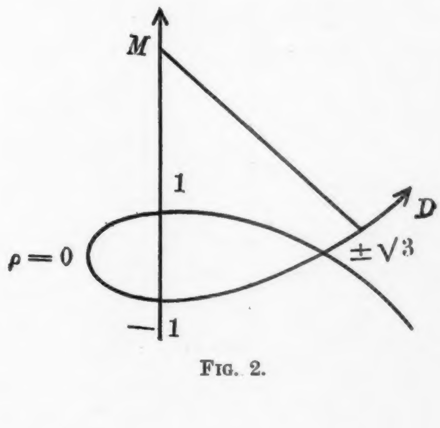


FIG. 2.

§ 3. *The elliptic case.*

When the dog is faster than the man, the relative curve is an ellipse. Write it, referred to a focus,

$$2x = \rho t + 1/\rho t + 2.$$

Giving x a vertical displacement ιs,

3.1)
$$2x = \rho t + 1/\rho t + 2 + 2\iota s,$$
$$2dx = dt(\rho - 1/\rho t^2) + 2\iota ds,$$

and $dx \perp x - \iota s$.

Hence

3.2)
$$\frac{dt(\rho^2 - 1/t^2) + 2\iota\rho ds}{-dt/t^2(\rho^2 - t^2) - 2\iota\rho ds} = -\left(\frac{\rho t + 1}{\rho + t}\right)^2,$$

$$2\iota\rho ds.(\rho^2 - 1)(t^2 - 1) = 1/t^2 \ (\rho t + 1)(\rho + t).2\rho(t^2 - 1)dt,$$

whence, $s = 0$, $t = 1$ being an assigned pair of values,

$$\iota s(\rho^2 - 1) = \rho t - \rho/t + (\rho^2 + 1)\log t.$$

Hence 3.2 becomes

3.3)
$$2(x - 1)(\rho^2 - 1)/(\rho^2 + 1) = \rho t - 1/\rho t + 2\log t.$$

Here $2\kappa = \rho + 1/\rho$.

The rectangular coördinates of the dog are then, with $t = \exp \iota\phi$,

3d)
$$X = \kappa\cos\phi + 1$$
$$Y = (\kappa\sin\phi + \phi)\,\kappa/\sqrt{(\kappa^2 - 1)},$$

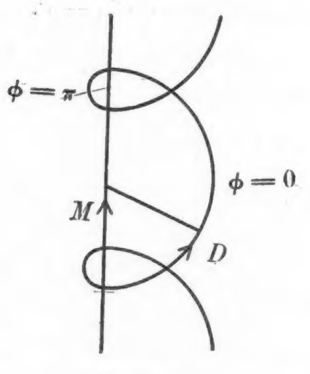


FIG. 3.

those of the man being

3m) $$X_1 = 0$$
$$Y_1 = s = (\sin\phi + \kappa\phi)/\sqrt{(\kappa^2 - 1)}.$$

The curve of ambience is symmetrical about any of the lines $Y = 0$, $\pm\pi, \cdots$

§ 4. *The hyperbolic case.*

Here we have $\kappa < 1$ and write

$$2\kappa = t + 1/t.$$

We have again the equation 3.1) but now ρ is the variable. Thus

4.1) $$2dx = d\rho(t - 1/\rho^2 t) + 2\iota ds,$$

and

4.2) $$\frac{d\rho(t - 1/\rho^2 t) + 2\iota ds}{d\rho(1/t - t/\rho^2) - 2\iota ds} = -\left(\frac{\rho t + 1}{\rho + t}\right)^2.$$

But this is precisely 3.2 with ρ and t interchanged. The algebraic reduction then leads to

$$\iota s(t^2 - 1 = t(\rho - 1/\rho) + (t^2 + 1)\log\rho$$

and

$$2(x - 1)(t^2 - 1)\ t^2 + 1) = \rho t - 1/\rho t + 2\log\rho,$$

or if $t = \exp\iota a$, so that $\kappa = \cos a$,

4.3) $$2\iota\tan a(x - 1) = (\rho - 1/\rho)\cos a + \iota(\rho + 1/\rho)\sin a + 2\log\rho.$$

We take ρ as positive only, so that for the right branch of the relative hyperbola $-\pi/2 < a < \pi/2$, for the left branch $\pi/2 < a < 3\pi/2$.

The rectangular coördinates of the dog are then

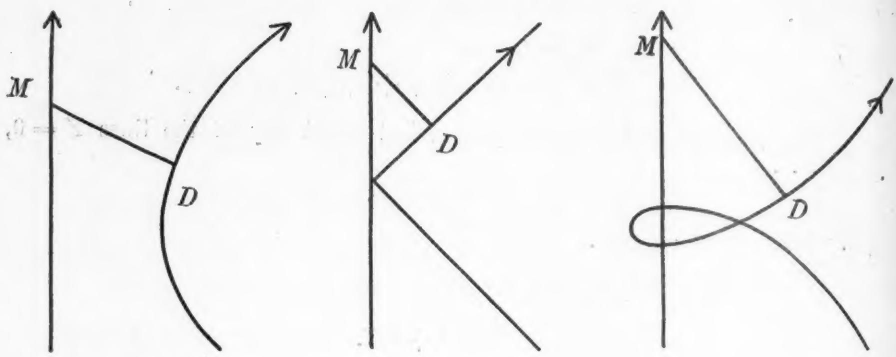
4d) $$X = \tfrac{1}{2}(\rho + 1/\rho)\cos a + 1,$$
$$Y = -\{\tfrac{1}{2}(\rho + 1/\rho)\cos a + \log\rho\}\cot a;$$

those of the man being

4m) $$X_1 = 0,$$
$$Y_1 = s = -\{\tfrac{1}{2}(\rho - 1/\rho) + \log\rho\cos a\}\csc a.$$

The right branch, for which $\cos a$ is positive, gives fig. 3a.

The left branch gives a curve which reflected in the axis $X = 0$ is indicated in fig. 3c.



3a 3b 3c

The separating case is that of two half lines. It is not immediately deducible from equation 4d, because they were formed from a proper hyperbola, with distinct foci. As the curves pass to ∞ the angles of the tangent tend to $\pm a$. But there are no asymptotes in the ordinary sense.

§ 5. *The relative curve for a circle.*

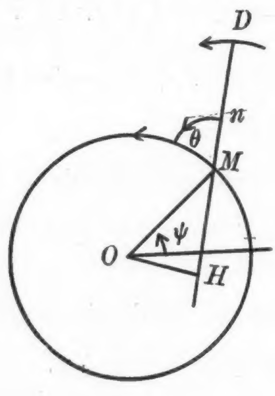


FIG. 4.

Let M describe a circle of radius ρ, and let its polar coördinates be ψ, ρ. Then $s = \rho\psi$, and (1.3) becomes

$$\frac{dn}{d\psi} = -\rho\cos\theta$$

$$= -OH, \text{ where } OH \text{ is } \perp MD.$$

$$= \frac{rdr}{ds_1}, \text{ where } r \text{ is } OD.$$

Since
$$ds_1 = \kappa ds = \kappa\rho d\psi,$$
$$rdr = \kappa\rho dn,$$

5.1) $$r^2 = 2\kappa\rho n + \lambda^2 \text{ when } \lambda \text{ is constant.}$$

This is the equation of the relative curve, the path of the dog when the man is brought to rest by imposing a rotation about O. It is the equation of a Cartesian, referred to a focus M and a pole or triple focus O. If in this curve the distances of a point from a focus be r_i and from the pole r, and the distance to the pole from a focus be δ_i $(i = 1, 2, 3)$ then

$$\sqrt{\delta_2\delta_3}\,r_1 - \delta_2\delta_3 = \cdot\cdot\cdot = \tfrac{1}{2}(r^2 - \Sigma\,\delta_2\delta_3).$$

(Harkness and Morley, *Theory of Functions,* p. 338).

The Cartesian is mapped on a line-segment by the equation

$$y = p(a + \iota\beta)$$

when either of the rectangular coördinates a, β is constant. Let β be constant. The foci are e_1, e_3, e_2, ∞. The pole or image of ∞ in the curve is the map of the image of $a = \beta = 0$ in the line $\beta = \text{const.}$ It is therefore $y = p2\iota\beta$. We take as standard case that in which the focus M is e_2, the left hand focus, and $2\beta < \omega_2/\iota - \beta$ so that the pole is outside the inner oval. With change of suffix the final formulae will apply to all cases.

The radius of the circle is now taken to be $e_2 - p2\iota\beta$ or $-\delta_2$. And the constants κ, λ are given by

5.2) $$\delta_1\delta_3/\delta_2{}^2 = \kappa^2, \quad \delta_2(\delta_1 + \delta_3) - \delta_1\delta_3 = \lambda^2\ .$$

§ 6. *The actual curve, for a circle.*

The Cartesian, referred to its pole, is

$$y = p(a + \iota\beta) - p2\iota\beta.$$

To this we apply the rotation

$$x = e^{\iota\psi}y,$$

and connect ψ and a, by requiring dx to be $\perp\ x - (e_2 - p2\iota\beta)e^{\iota\psi}$. That is,

$$\iota d\psi e^{\iota\psi}y + e^{\iota\psi}\dot{p}(a + \iota\beta)\,da \perp e^{\iota\psi}(p(a + \iota\beta) - e_2),$$

$$\frac{\iota y d\psi + \dot{p}(a + \iota\beta)\,da}{-\iota d\bar{y}\psi + \dot{p}(a - \iota\beta)\,da} = -\frac{p(a + \iota\beta) - e_2}{p(a - \iota\beta) - e_2}\,,$$

6.1) $$da\{\dot{p}(a + \iota\beta)(p(a - \iota\beta) - e_2) + \text{conj.}\}$$
$$+\,\iota d\psi\{p(a + \iota\beta) - p(a - \iota\beta)\}(p2\iota\beta - e_2) = 0.$$

We have then $\iota\delta_2 d\psi/da$ expressed as an elliptic function of a which has the simple poles $a = \iota\beta, -\iota\beta$, and no others. It is then

$$A\{\zeta(a+\iota\beta) - \zeta(a-\iota\beta)\} + B$$

and writing $a = \iota\beta + \epsilon$ and developing in powers of ϵ we have

6.2) $$\iota d\psi/da = 2\zeta(a+\iota\beta) - 2\zeta(a+\iota\beta) - c$$

where $$c = +2\zeta 2\iota\beta + \dot{p}2\iota\beta/(p2\iota\beta - e_2).$$

Hence $\iota\psi = 2\log\sigma(a+\iota\beta) - 2\log\sigma(a-\iota\beta) - ca$,

6m) $$\exp\iota\psi = \frac{\sigma^2(a+\iota\beta)}{\sigma^2(a-\iota\beta)}\exp - ca,$$

the constant of integration being assigned so that ψ and a vanish together.

Thus the actual path is mapped on the real axis by

$$x = \{p(a+\iota\beta) - p2\iota\beta\}\exp\iota\psi$$

$$= -\frac{\sigma(a+3\iota\beta)\sigma(a-\iota\beta)}{\sigma^2(a+\iota\beta)\sigma^2 2\iota\beta}\exp\iota\psi,$$

or finally

6d) $$x\sigma^2 2\iota\beta = -\frac{\sigma(a+3\iota\beta)}{\sigma(a-\iota\beta)}\exp - a\left[2\zeta 2\iota\beta + \frac{\dot{p}2\iota\beta}{p2\iota\beta - e_2}\right].$$

When a increases by the real period $2\omega_1$, x acquires the factor

$$\exp\left[8\iota\eta_1\beta - 2\omega_1 c\right].$$

Thus the curve is closed when the exponent is $\pi\iota \times$ a rational number.

On a Class of Invariant Subgroups of the Conformal and Projective Groups in Function Space.*

By I. A. Barnett.

Kowalewski has shown † that all the regular infinitesimal transformations which leave invariant the angle between two curves in function space, are given by

$$(1)\quad \delta f(x) = [\alpha(x) + cf(x) + \int_0^1 \beta(x,y) f(y)\,dy + f(x)\int_0^1 \epsilon(y) f(y)\,dy - \tfrac{1}{2}\epsilon(x)\int_0^1 f^2(y)\,dy]\,\delta t,$$

where α, β, ϵ are arbitrary continuous functions of their arguments, c is an arbitrary constant and $\beta(x,y) + \beta(y,x) = 0$. He shows furthermore that these transformations form a group, i. e., if δf_1 and δf_2 have the form (1), then the commutator ‡

$$(\delta f_1 ; \delta f_2) \equiv (\delta f_1, \delta f_2) - (\delta f_2, \delta f_1)$$

also has the form (1). He calls this the *conformal group* of function space. In the same paper he derives also the regular infinitesimal transformations which take every straight line of function space into another straight line. This he calls the *projective group* of function space and he finds that it has the form

$$(2)\quad \delta f(x) = [\alpha(x) + \beta(x) f(x) + \int_0^1 \gamma(x,y) f(y)\,dy + f(x)\int_0^1 \epsilon(y) f(y)\,dy]\,\delta t,$$

where α, β, γ, ϵ are arbitrary continuous functions of their arguments.

The object of this paper is to study the subgroups of (1) and (2), which leave invariant the manifold in function space

$$(3)\quad \int_0^1 \frac{f^n(x)}{\kappa(x)}\,dx = 1,$$

where $\kappa(x)$ is a non-vanishing continuous function on the interval. In other words, it is desired to find the explicit form of the functions α, β, etc., such

* Presented to the American Mathematical Society, April 13, 1923.

† G. Kowalewski, "Über Funktionraume II Mitteilung 9," *Sitzungsberichte der matematisch-naturwissenschaftlichen Klasse der Kaiserlichen Akademie der Wissenschaften zu Wien*, Vol. 120.

‡ For notations, see the paper already cited.

that the transformations (1) and (2) leave (3) invariant. After deriving the necessary condition that these functions must satisfy (§ 1), one is able to study explicitly the cases $n=1$ and $n=2$ (§§ 2, 3, 4, 5). Finally, it is shown (§ 6) that for $n > 2$ there are no subgroups of (1) and (2) which leave (3) invariant. Unless otherwise specified, all the integrations are from 0 to 1.

§ 1. *A Necessary Condition.*

In order that the infinitesimal transformations (1) leave invariant the manifold (3), one must have

$$\delta \int \frac{f^n(x)}{\kappa(x)}\, dx = 0 \qquad \text{when} \qquad \int \frac{f^n(x)}{\kappa(x)}\, dx = 1.$$

Substituting (1) in this, and making use of (3), one finds that

$$\begin{aligned} \int \frac{f^{n-1}(x)\alpha(x)}{\kappa(x)}\, dx + c + \int\int \frac{f^{n-1}(x) f(y) \beta(x,y)}{\kappa(x)}\, dxdy \\ + \int \epsilon(y) f(y)\, dy - \tfrac{1}{2} \int \frac{f^{n-1}(x)\epsilon(x)}{\kappa(x)}\, dx \int f^2(y)\, dy = 0 \end{aligned} \tag{4}$$

must be an identity for all f's which satisfy (3).

Set

$$f(x) = \frac{\phi(x)}{\left(\int \frac{\phi^n(x)}{\kappa(x)}\, dx\right)^{1/n}},$$

so that

$$\int \frac{f^n(x)}{\kappa(x)}\, dx = 1.$$

Then (4) becomes

$$\begin{aligned} \left(\int \frac{\phi^n(x)}{\kappa(x)}\, dx\right)^{2/n} \int \frac{\phi^{n-1}(x)\alpha(x)}{\kappa(x)}\, dx + c \left(\int \frac{\phi^n(x)}{\kappa(x)}\, dx\right)^{\frac{n+1}{n}} \\ + \int\int \frac{\beta(x,y)\phi^{n-1}(x)\phi(y)}{\kappa(x)}\, dxdy \left(\int \frac{\phi^n(x)}{\kappa(x)}\right)^{1/n} \\ + \int \epsilon(x)\phi(x) \int \frac{\phi^n(x)}{\kappa(x)}\, dx - \tfrac{1}{2} \int \frac{\phi^{n-1}(x)}{\kappa(x)}\, \epsilon(x)\, dx \int \phi^2(x)\, dx = 0, \end{aligned} \tag{5}$$

which must be an identity for all continuous functions $\phi(x)$.

If one carries through the analogous discussion for the transformations (2), one finds the identity

$$\begin{aligned} \left(\int \frac{\phi^n(x)}{\kappa(x)}\, dx\right)^{1/n} \int \frac{\alpha(x)\phi^{n-1}(x)}{\kappa(x)}\, dx + \int \frac{\beta(x)\phi^n(x)}{\kappa(x)}\, dx \\ + \int\int \frac{\gamma(x,y)\phi^{n-1}(x)\phi(y)}{\kappa(x)}\, dxdy + \int \epsilon(x)\phi(x)\, dx \left(\int \frac{\phi^n(x)}{\kappa(x)}\, dx\right)^{\frac{n-1}{n}} = 0 \end{aligned} \tag{6}$$

which must also hold for all continuous functions $\phi(x)$.

§ 2. *The Case $n = 1$ for the Conformal Group.*

The identity (5) reduces to

$$\left(\int \frac{\phi(x)}{\kappa(x)} dx\right)^2 \int_0^{\prime} \frac{\alpha(x)}{\kappa(x)} dx + c \left(\int \frac{\phi(x)}{\kappa(x)} dx\right)^2 +$$

$$\iint \frac{\beta(x,y)\phi(y)\,dy dx}{\kappa(x)} \int \frac{\phi(x)}{\kappa(x)} dx \;+ \int \epsilon(x)\phi(x)\,dx \int \frac{\phi(x)}{\kappa(x)} dx$$

$$-\tfrac{1}{2} \int \frac{\epsilon(x)}{\kappa(x)} dx \int \phi^2(x)\,dx = 0. \tag{7}$$

Choose $\phi(x)$ so that $\int \phi(x)\,dx/\kappa(x) = 0$, e. g., take $\phi(x) = \kappa(x) \sin 2\pi x$. Then (7) becomes

$$\int \frac{\epsilon(x)}{\kappa(x)} dx \int \kappa^2 \sin^2 2\pi x dx = 0$$

and since $\kappa(x) \neq 0$ on $(0, 1)$, it follows that $\int \epsilon(x)\,dx/\kappa(x) = 0$. On dividing by $\int \phi(x)\,dx/\kappa(x)$ one finds that equation (7) can be put in the form

$$\int \phi(y)\,dy \left[\frac{1}{\kappa(y)} \int \frac{\alpha(x)}{\kappa(x)} dx + \frac{c}{\kappa(y)} + \int \frac{\beta(x,y)}{\kappa(x)} dx + \epsilon(y)\right] = 0.$$

Since this must hold for all continuous functions $\phi(x)$ it follows that

$$\int \frac{\alpha(x)}{\kappa(x)} \; dx + c + \kappa(y) \int \frac{\beta(\xi, y)}{\kappa(\xi)} d\xi + \kappa(y)\epsilon(y) = 0.$$

Hence,

$$\kappa(y) \int \frac{\beta(\xi, y)}{\kappa(\xi)} \; d\xi + \kappa(y)\epsilon(y) = \text{const.}$$

$$\int \frac{\alpha(x)}{\kappa(x)} + c \qquad\qquad = - \text{const.}$$

From the first of these one obtains

$$\frac{1}{\kappa(y)} \int \frac{\beta(\xi, y)}{\kappa(\xi)} d\xi + \frac{\epsilon(y)}{\kappa(y)} = \frac{\text{const.}}{\kappa^2(y)} .$$

Integrating both sides with respect to y one has

$$\iint \frac{\beta(\xi, y)}{\kappa(\xi)\kappa(y)} d\xi dy + \int \frac{\epsilon(y)}{\kappa(y)} dy = \text{const.} \int \frac{dy}{\kappa^2(y)} dy.$$

But $\int \epsilon(y)\,dy/\kappa(y) = 0$, so that

$$\iint \frac{\beta(\xi, y)}{\kappa(\xi)\kappa(y)} d\xi dy = const. \int \frac{dy}{\kappa^2(y)}$$

Writing the last equation in the form

$$\frac{1}{2}\int\int\frac{\beta(\xi,y)}{\kappa(\xi)\kappa(y)}\,d\xi dy+\tfrac{1}{2}\int\int\frac{\beta(y,\xi)}{\kappa(\xi)\kappa(y)}\,d\xi dy=\text{const.}\int\frac{dy}{\kappa^2(y)}\,,$$

and remembering that $\beta(x,y)+\beta(y,x)=0$, one finds that the constant equals zero. Therefore,

$$c=-\int\frac{a(x)}{\kappa(x)}\,dx,$$

$$\epsilon(x)=-\int\frac{\beta(\xi,x)}{\kappa(\xi)}\,d\xi.$$

One has thus proved the result,

In order that the conformal group (1) *leave invariant the manifold in function space*

$$\int\frac{f(x)}{\kappa(x)}\,dx=1 \tag{8}$$

it must have the form

$$\begin{aligned}\delta f(x)&=[a(x)-f(x)\int\frac{a(\xi)}{\kappa(\xi)}\,d\xi+\int\beta(x,\xi)f(\xi)\,d\xi\\&=f(x)\int\int\frac{\beta(\xi,\eta)}{\kappa(\xi)}\,f(\eta)\,d\xi d\eta\\&\quad+\tfrac{1}{2}\int\frac{\beta(\xi,x)}{\kappa(\xi)}\,d\xi\int f^2(\xi)\,d\xi]\delta t,\quad \beta(x,y)+\beta(y,x)=0.\end{aligned} \tag{9}$$

It can be verified readily that when an infinitesimal transformation has the form (9), it will leave invariant the manifold (8). For,

$$\begin{aligned}\frac{1}{\delta t}\int\frac{\delta f(x)}{\kappa(x)}\,dx&=\left(\int\frac{a(x)}{\kappa(x)}\,dx-\int\frac{f(x)}{\kappa(x)}\,dx\int\frac{a(x)}{\kappa(x)}\,dx\right)\\&+\left(\int\int\frac{\beta(x,y)f(x)}{\kappa(x)}\,dxdy-\int\frac{f(x)}{\kappa(x)}\,dx\int\int\frac{\beta(x,y)f(y)}{\kappa(y)\kappa(x)}\,dxdy\right)\\&+\tfrac{1}{2}\int\int\frac{\beta(y,x)}{\kappa(x)\kappa(y)}\,dxdy\int f^2(y)\,dy.\end{aligned}$$

But, since $\int f(x)\,dx/\kappa(x)=1$ the first two parentheses are zero and since $\beta(x,y)+\beta(y,x)=0$, the last term vanishes.

It will now be shown that (9) possesses the group property. One finds after considerable computation

$$\left(\alpha_1(x) - f(x)\int\frac{\alpha_1(\xi)}{\kappa(\xi)}\,d\xi;\quad \alpha_2(x) - f(x)\int\frac{\alpha_2(\xi)}{\kappa(\xi)}\,d\xi\right)$$
$$= \alpha(x) - f(x)\int\frac{\alpha(\xi)}{\kappa(\xi)}\,d\xi,$$

$$\Big(\int\beta_1(x,\xi)f(\xi)d\xi - f(x)\int\int\frac{\beta_1(\xi,\eta)f(\eta)}{\kappa(\xi)}\,d\xi d\eta$$
$$+\tfrac{1}{2}\int\frac{\beta_1(\xi,x)}{\kappa(\xi)}\,d\xi\int f^2(\xi)d\xi;$$

$$\int\beta_2(x,\xi)f(\xi)d\xi - f(x)\int\int\frac{\beta_2(\xi,\eta)f(\eta)}{\kappa(\xi)}\,d\xi d\eta + \tfrac{1}{2}\int\frac{\beta_2(\xi,x)}{\kappa(\xi)}d\xi\int f^2(\xi)d\xi\Big)$$
$$= \int\gamma(x,\xi)f(\xi)d\xi - f(x)\int\int\frac{\gamma(\xi,\eta)}{\kappa(\xi)}\,f(\eta)d\xi d\eta$$
$$+\tfrac{1}{2}\int\frac{\gamma(\xi,x)}{\kappa(\xi)}\,d\xi\int f^2(\xi)d\xi;$$

$$\Big(\alpha_1(x) - f(x)\int\frac{\alpha_1(\xi)}{\kappa(\xi)}\,d\xi;\quad \int\beta_1(x,\xi)f(\xi)d\xi - f(x)\int\int\frac{\beta_1(\xi,\eta)}{\kappa(\xi)}\,f(\eta)d\xi d\eta$$
$$+\tfrac{1}{2}\int\frac{\beta_1(\xi,x)}{\kappa(\xi)}\,d\xi\int f^2(\xi)d\xi\Big)$$
$$= \bar{\alpha}(x) - f(x)\int\frac{\bar{\alpha}(\xi)}{\kappa(\xi)}\,d\xi + \int\bar{\beta}(x,\xi)f(\xi)d\xi - f(x)\int\int\frac{\bar{\beta}(\xi,\eta)}{\kappa(\xi)}\,f(\eta)d\xi d\eta$$
$$+\tfrac{1}{2}\int\frac{\bar{\beta}(\xi,x)}{\kappa(\xi)}\,d\xi\int f^2(\xi)d\xi,$$

where

$$\alpha(x) = \alpha_1(x)\int\frac{\alpha_2(\xi)}{\kappa(\xi)}\,d\xi - \alpha_2(x)\int\frac{\alpha_1(\xi)}{\kappa(\xi)}\,d\xi,$$
$$\gamma(x,y) = \int\{\beta_1(x,\xi)\beta_2(\xi,y) - \beta_2(x,\xi)\beta_1(\xi,y)\}d\xi,$$
$$\bar{\alpha}(x) = -\int\beta_1(x,\xi)\alpha_1(\xi)d\xi,$$
$$\bar{\beta}(x,y) = \alpha(x)\int\frac{\beta_1(\eta,y)}{\kappa(\eta)}\,d\eta - \alpha(y)\int\frac{\beta_1(\eta,x)}{\kappa(\eta)}\,d\eta.$$

It is to be noticed that $f(x)\int\alpha(x)dx/\kappa(x) = 0$ since $\int\alpha(x)dx/\kappa(x) = 0$ and that $\gamma(x,y) + \gamma(y,x) = 0$ because $\beta_1(x,y)$ and $\beta_2(x,y)$ have the same property. Since it is obvious that $\bar{\beta}(x,y)$ has the property $\bar{\beta}(x,y) + \bar{\beta}(y,x) = 0$ one has verified that the transformations (9) form a group.

In particular, when $\kappa(x) \equiv 1$ it is seen that the conformal infinitesimal transformations which leave the area under the curve invariant, are given by

$$\delta f(x) = [a(x) - f(x)\int a(\xi)\,d\xi + \int \beta(x,y) f(y)\,dy - f(x)\int\int \beta(y,z) f(y)\,dy\,dz + \tfrac{1}{2}\int \beta(y,\xi)\,d\xi \int f^2(\xi)\,d\xi]\,\delta t.$$

§ 3. *The Case $n = 2$ for the Conformal Group.*

In this case equation (5) reduces to

$$\int \frac{\phi^2(x)}{\kappa(x)}\,dx \int \frac{\phi(x)a(x)}{\kappa(x)}\,dx + c\left\{\int \frac{\phi^2(x)}{\kappa(x)}\,dx\right\}^{3/2} + \int\int \frac{\beta(x,y)\phi(x)\phi(y)}{\kappa(x)}\,dx\,dy\left(\int \frac{\phi^2(x)}{\kappa(x)}\,dx\right)^{1/2} + \int \epsilon(x)\phi(x)\,dx \int \frac{\phi^2(x)}{\kappa(x)}\,dx - \tfrac{1}{2}\int \frac{\phi(x)\epsilon(x)}{\kappa(x)}\,dx \int \phi^2(x)\,dx = 0. \tag{10}$$

Choose the function $\phi(x)$ so that

$$\phi(x) > 0, \quad x_1 - \delta \leqq x \leqq x_1 + \delta$$
$$\phi(x) \equiv 0, \quad x > x_1 + \delta, \quad x < x_1 - \delta$$

where x_1 is a definite value of the variable x in the interval $(0, 1)$.

Using this function in (10) and dividing through by $\left(\int \frac{\phi^2(x)}{\kappa(x)}\,dx\right)^{3/2}$ one finds that equation (10) may be written

$$\frac{\int_{x_1-\delta}^{x_1+\delta} \frac{\phi(x)a(x)}{\kappa(x)}\,dx}{\left(\int_{x_1-\delta}^{x_1+\delta} \frac{\phi^2(x)}{\kappa(x)}\,dx\right)^{1/2}} + c + \frac{\int_{x_1-\delta}^{x_1+\delta}\int_{x_1-\delta}^{x_1+\delta} \frac{\beta(x,y)\phi(x)\phi(y)}{\kappa(x)}\,dx\,dy}{\int_{x_1-\delta}^{x_1+\delta} \frac{\phi^2(x)}{\kappa(x)}\,dx}$$
$$+ \frac{\int_{x_1-\delta}^{x_1+\delta} \epsilon(x)\phi(x)\,dx}{\left(\int_{x_1-\delta}^{x_1+\delta} \frac{\phi^2(x)}{\kappa(x)}\,dx\right)^{1/2}} - \tfrac{1}{2}\int_{x_1-\delta}^{x_1+\delta} \frac{\phi(x)\epsilon(x)}{\kappa(x)}\,dx\; \frac{\int_{x_1-\delta}^{x_1+\delta} \phi^2(x)\,dx}{\left(\int_{x_1-\delta}^{x_1+\delta} \frac{\phi^2(x)}{\kappa(x)}\,dx\right)^{3/2}} = 0.$$

On applying the Mean Value Theorem of Definite Integrals and Schwarz's Inequality

$$\left(\int_{x_1-\delta}^{x_1+\delta} \frac{\phi(x)}{\sqrt{\kappa(x)}}\,dx\right)^2 \leqq 2\delta \int_{x_1-\delta}^{x_1+\delta} \frac{\phi^2(x)}{\kappa(x)}\,dx$$

to the left hand member of the preceding equation, one obtains, after passing to the limit as $\delta \to 0$ that $c = 0$.

Equation (10) may be put in the form

$$\int\int \phi(x)\phi(y)\,dx\,dy\left[\frac{a(x)\phi(y)}{\kappa(x)\kappa(y)} + \frac{\beta(x,y)\left(\int \phi^2(x)\,dx\right)^{1/2}}{\kappa(x)} + \frac{\epsilon(x)\phi(y)}{\kappa(y)} - \tfrac{1}{2}\frac{\epsilon(x)\phi(y)}{\kappa(x)}\right] = 0.$$

Now, it is easy to show * that if

$$\int'_0 \int'_0 \theta(x,y)\phi(x)\phi(y)\,dxdy = 0$$

for all continuous functions $\phi(x)$ one must have $\theta(x,y)+\theta(y,x)=0$. Hence,

$$(11)\quad \phi(y)\left[\frac{a(x)}{\kappa(x)\kappa(y)}+\frac{\epsilon(x)}{\kappa(y)}-\tfrac{1}{2}\frac{\epsilon(x)}{\kappa(x)}\right]+\phi(x)\left[\frac{a(y)}{\kappa(x)\kappa(y)}+\frac{\epsilon(y)}{\kappa(x)} - \tfrac{1}{2}\frac{\epsilon(y)}{\kappa(y)}\right]+[\int\phi^2(x)\,dx]^{\frac{1}{2}}\left[\frac{\beta(x,y)\kappa(y)+\beta(y,x)\kappa(x)}{\kappa(x)\kappa(y)}\right]=0,$$

must be an identity for all x, y of the square $0 \leqq x \leqq 1$, $0 \leqq y \leqq 1$. Consider now a function $\phi(x)$ and two fixed values x, y which satisfy the conditions 1) $\phi(y)=0$, 2) $\phi(x)=l$ where l is an arbitrary constant, 3) $\phi(x)$ arbitrary and continuous on the rest of the interval $(0,1)$. Using this function in equation (11), one finds that the latter becomes

$$(12)\quad l\left[\frac{a(y)}{\kappa(x)\kappa(y)}+\frac{\epsilon(y)}{\kappa(x)}-\tfrac{1}{2}\frac{\epsilon(y)}{\kappa(y)}\right] + [\int\phi^2(x)\,dx]^{\frac{1}{2}}\left[\frac{\beta(x,y)\kappa(y)+\beta(y,x)\kappa(x)}{\kappa(x)\kappa(y)}\right]=0,$$

and since l is arbitrary,

$$\frac{a(y)}{\kappa(x)\kappa(y)}+\frac{\epsilon(y)}{\kappa(x)}-\tfrac{1}{2}\frac{\epsilon(y)}{\kappa(y)}=0,$$

so that

$$\kappa(x)=2/\epsilon(y)\,[a(y)+\epsilon(y)\kappa(y)].$$

Therefore, $\kappa(x)=\text{const.}=k$ and $a(y)+k.\epsilon(y)=k\epsilon(y)/2$ or $a(y)=-k/2\ \epsilon(y)$. Also from (12), one has $\beta(x,y)\kappa(y)+\beta(y,x)\kappa(x)\equiv 0$ so that $\beta(x,y)+\beta(y,x)\equiv 0$ since $\kappa(y)=k$ is constant. Hence, there is no additional restriction upon the function $\beta(x,y)$. Thus one has proved the result.

There are no subgroups of the conformal group which leave

$$\int\frac{f^2(x)}{\kappa(x)}\,dx=1$$

* Cf. Kowalewski, "Über Funktionräume," I Mitteilung § 8, Vol. 120 of the Journal already mentioned. Kowalewski asserts that if $\int\int\theta(x,y)\phi(x)\phi(y)\,dxdy=0$ for all continuous functions $\phi(x)$ then $\theta(x,y)\equiv 0$ which is obviously false since any skew-symmetric function will satisfy the preceding equation. In fact his argument tacitly assumes that $Q(x,y)=Q(y,x)$ in which case his conclusion is correct.

invariant, unless $\kappa(x)$ is a constant k in which case the infinitesimal transformations have the form

$$(13)\quad \delta f(x) = [-\tfrac{1}{2}k.\epsilon(x) + \int\beta(x,y)f(y)\,dy + f(x)\int\epsilon(y)f(y)\,dy - \tfrac{1}{2}\epsilon(x)\int f^2(y)\,dy]\delta t.$$

One can readily verify that $\int f(x)\delta f(x) = 0$ when $\int f^2(x)\,dx = k$. To show that the transformations (13) form a group, one verifies that

$$(\int\beta_1(x,y)f(y)\,dy;\ \int\beta_2(x,y)f(y)\,dy) = \int\beta(x,y)f(y)\,dy,$$

$$(-\tfrac{1}{2}k\epsilon_1(x) + f(x)\int\epsilon_1(y)f(y)\,dy - \tfrac{1}{2}\epsilon_1(x)\int f^2(y)\,dy;\ -\tfrac{1}{2}k\epsilon_2(x) + f(x)\int\epsilon_2(y)f(y)\,dy - \tfrac{1}{2}\epsilon(x)\int f^2(y)\,dy) = \int\bar\beta(x,y)f(y)\,dy,$$

$$(\int\beta_3(x,y)f(y)\,dy;\ -\tfrac{1}{2}k\epsilon_3(x) + f(x)\int\epsilon_3(y)f(y)\,dy - \tfrac{1}{2}\epsilon_3(x)\int f^2(y)\,dy) = -\tfrac{1}{2}k.\epsilon(x) + f(x)\int\epsilon(y)f(y)\,dy - \tfrac{1}{2}\epsilon(x)\int f^2(y)\,dy,$$

where

$$\beta(x,y) = \int[\beta_1(x,\xi)\beta_2(\xi,y) - \beta_2(x,\xi)\beta_1(\xi,y)]\,d\xi,$$
$$\bar\beta(x,y) = k[\int\epsilon_1(x)\epsilon_2(y) - \epsilon_2(x)\epsilon_1(y)],$$
$$\epsilon(x) = \int\beta(x,\xi)\epsilon(\xi)\,d\xi.$$

Since

$$\beta(x,y) + \beta(y,x) = 0$$
$$\bar\beta(x,y) + \bar\beta(y,x) = 0,$$

one has verified the group property of (13).

In particular, when $k = 1/\pi$, it is seen that the conformal transformations which leave invariant the volume of the surface of revolution generated by $f(x)$, are given by

$$\delta f(x) = [-\tfrac{1}{2}\epsilon(x) + \int\beta(x,y)f(y)\,dy + f(x)\int\epsilon(y)f(y)\,dy - \tfrac{1}{2}\epsilon(x)\int f^2(y)\,dy]\delta t.$$

§ 4. *The Case $n = 1$ for the Projective Group.*

When $n = 1$, equation (6) becomes

$$\int\frac{\phi(x)}{\kappa(x)}\,dx\int\frac{\alpha(x)}{\kappa(x)}\,dx + \int\frac{\beta(x)}{\kappa(x)}\phi(x)\,dx + \int\int\frac{\gamma(x,y)\phi(y)}{\kappa(x)}\,dx\,dy + \int\epsilon(x)\phi(x)\,dx = 0,$$

so that

$$\int\phi(y)\,dy\Big[\int\frac{\alpha(x)}{\kappa(x)\kappa(y)}\,dx + \frac{\beta(y)}{\kappa(y)} + \int\frac{\gamma(x,y)}{\kappa(x)}\,dx + \epsilon(y)\Big] = 0$$

must be an identity for all continuous functions $\phi(x)$. Hence, for all (x, y),

$$\frac{1}{\kappa(y)}\int\frac{\alpha(x)}{\kappa(x)}dx + \frac{\beta(y)}{\kappa(y)} + \int\frac{\gamma(x,y)}{\kappa(x)}dx + \epsilon(y) = 0,$$

from which follows that

$$\epsilon(y) = -\frac{1}{\kappa(y)}\left[\beta(y) + \int\frac{\alpha(x)}{\kappa(x)}dx\right] - \int\frac{\gamma(x,y)}{\kappa(x)}dx.$$

Thus, *all the projective infinitestimal transformations which leave* $\int f(x)dx/\kappa(x) = 1$ *invariant are given by*

$$\begin{aligned}\delta f(x) = [\alpha(x) - f(x)\int\frac{\alpha(\xi)}{\kappa(\xi)}d\xi + \beta(x)f(x) - f(x)\int\frac{\beta(y)f(y)}{\kappa(y)}dy \\ + \int\gamma(x,y)f(y)dy - f(x)\int\int\frac{\gamma(x,y)}{\kappa(x)}f(y)\,dxdy]\delta t.\end{aligned} \tag{14}$$

One finds after some computation that

$$\begin{aligned}\left(\alpha_1(x) - f(x)\int\frac{\alpha_1(y)}{\kappa(y)}dy;\ \alpha_2(x) - f(x)\int\frac{\alpha_2(y)}{\kappa(y)}dy\right) \\ = \alpha(x) - f(x)\int\frac{\alpha(y)}{\kappa(y)}dy,\end{aligned}$$

$$\left(\beta_1(x)f(x) - f(x)\int\frac{\beta_1(y)f(y)}{\kappa(y)}dy;\ \beta_2(x)f(x) - f(x)\int\frac{\beta_2(y)f(y)}{\kappa(y)}dy\right) = 0,$$

$$\begin{aligned}\Big(\int\gamma_1(x,y)f(y)dy - f(x)\int\int\frac{\gamma_1(x,y)}{\kappa(x)}f(y)\,dxdy;\ \int\gamma_2(x,y)f(y)dy \\ - f(x)\int\int\frac{\gamma_2(x,y)}{\kappa(x)}f(y)\,dxdy\Big) = \int\gamma(x,y)f(y)dy \\ - f(x)\int\int\frac{\gamma(x,y)f(y)}{\kappa(x)}dxdy,\end{aligned}$$

$$\begin{aligned}\left(\alpha_1(x) - f(x)\int\frac{\alpha_1(y)}{\kappa(y)}dy;\ \beta_1(x)f(x) - f(x)\int\frac{\beta_1(y)f(y)}{\kappa(y)}dy\right) \\ = \alpha'(x) - f(x)\frac{\alpha'(y)}{\kappa(y)}dy + \int\gamma'(x,y)f(y)dy - f(x)\int\int\frac{\gamma'(x,y)}{\kappa(x)}f(y)\,dxdy,\end{aligned}$$

$$\begin{aligned}\left(\alpha_1(x) - f(x)\int\frac{\alpha_1(y)}{\kappa(y)}dy;\ \int\gamma_1(x,y)f(y)dy - f(x)\int\int\frac{\gamma_1(x,y)f(y)}{\kappa(x)}dxdy\right) \\ = \alpha''(x) - f(x)\int\frac{\alpha''(y)}{\kappa(y)}dy + \int\gamma''(x,y)f(y)dy - f(x)\int\int\frac{\gamma''(x,y)}{\kappa(x)}f(y)\,dxdy,\end{aligned}$$

$$(\beta_1(x)f(x) - f(x)\int\frac{\beta_1(y)f(y)}{\kappa(y)}\,dy;\ \int\gamma_1(x,y)f(y)\,dy$$
$$-f(x)\int\int\frac{\gamma_1(x,y)f(y)}{\kappa(x)}\,dxdy) = \int\gamma'''(x,y)f(y)\,dy$$
$$-f(x)\int\int\frac{\gamma'''(x,y)f(y)}{\kappa(x)}\,dxdy,$$

where

$$a(x) = a_1(x)\int\frac{a_2(y)}{\kappa(y)}\,dy - a_2(x)\int\frac{a_1(y)}{\kappa(y)}\,dy,$$
$$\gamma(x,y) = \int[\gamma_1(x,\xi)\gamma_2(\xi,y) - \gamma_2(x,\xi)\gamma_1(\xi,y)]\,d\xi,$$
$$a'(x) = -\beta_1(x)a_1(x),\quad \gamma'(x,y) = \frac{a_1(x)\beta_1(y)}{\kappa(y)},$$
$$a''(x) = \int\gamma_1(x,\xi)a_1(\xi)\,d\xi,\quad \gamma''(x,y) = a_1(x)\int\frac{\gamma_1(x,y)}{\kappa(x)}\,dx,$$
$$\gamma'''(x) = \beta_1(x)\gamma_1(x,y) - \gamma_1(x,y)\beta_1(y).$$

This proves that the transformations (14) have the group property.

§ 5. *The Case $n = 2$ for the Projective Group.*

Equation (6) reduces in this case to

$$\left(\int\frac{\phi^2(x)}{\kappa(x)}\,dx\right)^{\frac{1}{2}}\int\frac{a(x)\phi(x)}{\kappa(x)}\,dx + \int\frac{\beta(x)\phi^2(x)}{\kappa(x)}\,dx$$
$$+\int\int\frac{\gamma(x,y)\phi(x)\phi(y)}{\kappa(x)}\,dxdy + \int\epsilon(x)\phi(x)\,dx\left(\int\frac{\phi^2(x)}{\kappa(x)}\,dx\right)^{\frac{1}{2}} = 0,$$

or

$$\left(\int\frac{\phi^2(x)}{\kappa(x)}\,dx\right)^{\frac{1}{2}}\left\{\int\left[\frac{a(x)}{\kappa(x)} + \epsilon(x)\right]\phi(x)\,dx\right\} + \int\frac{\beta(x)\phi^2(x)}{\kappa(x)}\,dx$$
$$(15)\qquad + \int\int\frac{\gamma(x,y)\phi(x)\phi(y)}{\kappa(x)}\,dxdy = 0.$$

Now, define the function $\phi(x)$ as follows,

$$(16)\qquad \phi(x) > 0,\quad x_1 - \delta \leqq x \leqq x_1 + \delta,$$
$$\phi(x) \equiv 0,\quad x > x_1 + \delta,\quad x < x_1 - \delta.$$

Applying this function to the left hand side of (15), and using the Mean Value Theorem of Definite Integrals, one finds after dividing through by $\dfrac{\phi^2(x)}{\kappa(x)}\,dx$ that

$$\frac{\displaystyle\int_{x_1-\delta}^{x_1+\delta}\left[\frac{a(x)}{\kappa(x)}+\epsilon(x)\right]\phi(x)\,dx}{\left(\displaystyle\int_{x_1-\delta}^{x_1+\delta}\frac{\phi^2(x)}{\kappa(x)}\,dx\right)^{1/2}}+\beta(\xi)+\frac{\gamma(\zeta,\eta)}{\kappa(\zeta)}\,\frac{\displaystyle\int_{x_1-\delta}^{x_1+\delta}\phi(x)\,dx}{\displaystyle\int_{x_1-\delta}^{x_1+\delta}\frac{\phi^2(x)}{\kappa(x)}\,dx}=0.$$

Here ξ, η, ζ are suitably chosen values of x in the interval $x_1-\delta \leqq x \leqq x_1+\delta$. If one now applies Schwarz's Inequality $\left(\int_{x_1-\delta}^{x_1+\delta}\phi(x)\,dx\right)^2 \leqq 2\delta\int_{x_1-\delta}^{x_1+\delta}\phi^2(x)\,dx$ and passes to the limit as $\delta\rightarrow 0$ one sees that the preceding equation reduces to $\beta(x_1)=0$. It must be remembered, of course, that in obtaining the limiting value, use is made of the continuity of the functions β and γ. Since x_1 was arbitrary, it follows that $\beta(x)\equiv 0$. Putting this in (15) and dividing by $\left(\int\frac{\phi^2(x)}{\kappa(x)}\,dx\right)^{1/2}$ one gets

$$\int_0^{\prime}\left[\frac{a(x)}{\kappa(x)}+\epsilon(x)\right]\phi(x)\,dx+\frac{\displaystyle\int_0^{\prime}\int_0^{\prime}\frac{\gamma(x,y)\phi(x)\phi(y)}{\kappa(x)}\,dx\,dy}{\left(\displaystyle\int_0^{\prime}\frac{\phi^2(x)}{\kappa(x)}\,dx\right)^{1/2}}=0.$$

Applying the function (16) to this last equation and proceeding as before, one finds readily that

$$\frac{a(x)}{\kappa(x)}+\epsilon(x)=0.$$

Hence,

$$\int\int\frac{\gamma(x,y)\phi(x)\phi(y)}{\kappa(x)}\,dx\,dy=0.$$

Let $\gamma(x,y)/\kappa(x)=\theta(x,y)$. Then it must follow, as has already been seen in another connection, that

$$\theta(x,y)+\theta(y,x)=0.$$

Thus, *all the projective infinitesimal transformations which leave invariant the manifold*

(17) $$\int\frac{f^2(x)}{\kappa(x)}\,dx=1$$

are given by

(18) $$\delta f(x)=\left[-\kappa(x)\epsilon(x)+\int\kappa(x)\theta(x,y)f(y)\,dy + f(x)\int\epsilon(y)f(y)\,dy\right]\delta t,$$

where $$\theta(x,y)+\theta(y,x)=0.$$

One can readily verify that these transformations do leave (17) invariant. For,

$$\frac{1}{\delta t}\int\frac{\delta f(x)}{\kappa(x)}\,dx=\int-\epsilon(x)f(x)\,dx+\int\int\theta(x,y)f(x)f(y)\,dxdy$$
$$+\int\frac{f^2(x)}{\kappa(x)}\,dx\int\epsilon(y)f(y)\,dy=0,$$

since $\int f^2(x)\,dx/\kappa(x)=1$ and $\theta(x,y)+\theta(y,x)=0$.

In particular, when $\kappa(x)\equiv 1$, one has the case already treated by Kowalewski (loc. cit. § 8).

To prove that the transformations (18) form a group one verifies that

$$\left(\int\kappa(x)\theta_1(x,y)f(y)\,dy;\ \int\kappa(x)\theta_2(x,y)f(y)\,dy\right)=\int\kappa(x)\theta(x,y)f(y)\,dy,$$
$$\left(-\kappa(x)\epsilon_1(x)+f(x)\int\epsilon_1(y)f(y)\,dy;\ -\kappa(x)\epsilon_2(x)+f(x)\int\epsilon_2(y)f(y)\,dy\right)$$
$$=\int\kappa(x)\bar{\theta}(x,y)f(y)\,dy,$$
$$\left(\int\kappa(x)\theta_1(x,y)f(y)\,dy;\ -\kappa(x)\epsilon_1(x)+f(x)\int\epsilon(y)f(y)\,dy\right)$$
$$=-\kappa(x)\bar{\epsilon}(x)+f(x)\int\bar{\epsilon}(y)f(y)\,dy,$$

where

$$\theta(x,y)=\int\{\theta_1(x,\xi)\theta_2(\xi,y)-\theta_2(x,\xi)\theta_1(\xi,y)\}\,d\xi,$$
$$\bar{\theta}(x,y)=\epsilon_1(x)\epsilon_2(y)-\epsilon_2(x)\epsilon_1(y),$$
$$\bar{\epsilon}(x)=\int\theta(x,y)\kappa(y)\epsilon(y)\,dy.$$

It is obvious from the form of $\bar{\theta}(x,y)$ that it has the property $\bar{\theta}(x,y)+\bar{\theta}(y,x)=0$. Hence the group property has been proved.

§ 6. *The Case $n>2$ for the Projective and Conformal Groups.*

Consider first the equation

$$(6)\qquad \left(\int\frac{\phi^n(x)}{\kappa(x)}\,dx\right)^{1/n}\cdot\int\frac{\alpha(x)\phi^{n-1}(x)}{\kappa(x)}\,dx+\int\frac{\beta(x)\phi^n(x)}{\kappa(x)}\,dx$$
$$+\int\int\frac{\gamma(x,y)\phi^{n-1}(x)\phi(y)}{\kappa(x)}\,dxdy\ +\int\epsilon(x)\phi(x)\,dx\left(\int\frac{\phi^n(x)}{\kappa(x)}\,dx\right)^{\frac{n-1}{n}}=0.$$

Define the function $\phi(x)$ as follows

$$(19)\qquad \begin{aligned}\phi(x)&=x-(x_1-\delta), && x_1-\delta\leqq x\leqq x_1,\\ &=-x+(x_1+\delta), && x_1\leqq x\leqq x_1+\delta,\\ &\equiv 0, && x>x_1+\delta,\ \ x_1<x_1-\delta.\end{aligned}$$

Using this function in (6) and applying the Mean Value Theorem of Definite Integrals, one has

$$\frac{a(\xi_1)}{\kappa(\xi_1)}\frac{1}{[\kappa(\xi_2)]^{1/n}}\int_{x_1-\delta}^{x_1+\delta}\phi^n(x)\,dx\int_{x_1-\delta}^{x_1+\delta}\phi^{n-1}(x)\,dx+\frac{\beta(\xi_2)}{\kappa(\xi_2)}\int_{x_1-\delta}^{x_1+\delta}\phi^n(x)\,dx$$
$$+\frac{\gamma(\xi_3,\eta_3)}{\kappa(\xi_3)}\int_{x_1-\delta}^{x_1+\delta}\phi^{n-1}(x)\,dx\int_{x_1-\delta}^{x_1+\delta}\phi(x)\,dx$$
$$+\frac{\epsilon(\xi_4)}{[\kappa(\xi_5)]^{1-1/n}}\int_{x_1-\delta}^{x_1+\delta}\phi(x)\,dx\left(\int_{x_1-\delta}^{x_1+\delta}\phi^n(x)\,dx\right)^{1-1/n}=0,$$

where $\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \eta_3$ are suitably chosen values in the interval $x_1-\delta\leqq x \leqq x_1+\delta$. Since

$$\int_{x_1-\delta}^{x_1+\delta}\phi^\kappa(x)\,dx=\int_{x_1-\delta}^{x_1}\phi^\kappa(x)\,dx+\int_{x_1}^{x_1+\delta}\phi^\kappa(x)\,dx=\int_{x_1-\delta}^{x_1}[x-(x_1-\delta)]^\kappa dx$$
$$+\int_{x_1}^{x_1+\delta}[-x+(x_1+\delta)]^\kappa dx=\frac{\delta^{\kappa+1}}{\kappa+1}+\frac{\delta^{\kappa+1}}{\kappa+1}=\frac{2\delta^{\kappa+1}}{\kappa+1},$$

one may write the preceding in the form

$$\frac{a(\xi_1)}{\kappa(\xi_1)}\frac{1}{[\kappa(\xi_2)]^{1/n}}\frac{2^{1+1/n}\delta^{n+1+1/n}}{n(n+1)}+\frac{\beta(\xi_2)}{\kappa(\xi_2)}\frac{2\delta^{n+1}}{n+1}+\frac{4\gamma(\xi_3,\eta_3)}{\kappa(\xi_3)}\frac{\delta^{n+2}}{2n}$$
$$+\frac{\epsilon(\xi_4)}{[\kappa(\xi_5)]^{1-1/n}}\frac{2^{2-1/n}\delta^{n+2-1/n}}{2(2-1/n)}=0.$$

Dividing through by δ^{n+1} and passing to the limit as $\delta\to 0$ one finds, since $n>1$ that $\beta(x_1)=0$. Proceeding in this manner one is able to prove, since $n>2$, that all the functions a, β and ϵ are identically zero. Hence, equation (6) reduces to $\int\int\gamma(x,y)\phi^{n-1}(x)\phi(y)\,dxdy/\kappa(x)=0$ from which it follows that $\gamma(x,y)\equiv 0$.

Thus it has been proved that *there are no subgroups of the projective group which leave* (3) *invariant* for $n>2$.

Consider now the equation

$$(5)\qquad \left(\int\frac{\phi^n(x)}{\kappa(x)}dx\right)^{2/n}\int\frac{\phi^{n-1}(x)\,dx}{\kappa(x)}+c\left(\int\frac{\phi^n(x)}{\kappa(x)}dx\right)^{1+1/n}$$
$$+\int\int\frac{\beta(x,y)\phi^{n-1}(x)\phi(y)}{\kappa(x)}dxdy\left(\int\frac{\phi^n(x)}{\kappa(x)}dx\right)^{1/n}$$
$$+\int\epsilon(x)\phi(x)\,dx\int\frac{\phi^n(x)}{\kappa(x)}dx-\tfrac{1}{2}\int\frac{\phi^{n-1}(x)}{\kappa(x)}\,\epsilon(x)\,dx\int\phi^2(x)\,dx=0.$$

If, as before, one applies the function $\phi(x)$ defined by (19) to this equation, one obtains after an application of the Mean Value Theorem,

$$\frac{2^{1+2/n}\alpha(\xi_2)}{\kappa(\xi_2)[\kappa(\xi_1)]^{2/n}}\,\delta^{n+2+2/n}+\frac{2^{1+1/n}.\,c.\,\delta^{n+2+1/n}}{[\kappa(\xi_3)]^{1+1/n}}+\frac{2^{2+1/n}\beta(\xi_4,\eta_4)}{\kappa(\xi_4)[\kappa(\xi_5)]^{1/n}}\,\delta^{n+3+1/n}$$
$$+\frac{4\epsilon(\xi_6)}{\kappa(\xi_7)}\,\delta^{n+3}-4\,\frac{\epsilon(\xi_8)}{2\kappa(\xi_8)}\,\delta^{n+3}=0,$$

where $\xi_1, \cdots \xi_8, \eta_4$ are suitably chosen values of x in the interval $x_1 - \delta \leqq x \leqq x_1 + \delta$. Dividing this last equation through by $\delta^{n+2+1/n}$ and passing to the limit as $\delta \to 0$ one finds that $c = 0$ since $n > 1$. In a similar fashion, using the fact that $n > 2$ one can prove that $\alpha(x) \equiv 0$ and $\epsilon(x) \equiv 0$. Substituting this in equation (5), one obtains $\int\int \beta(x,y)\phi^{n-1}(x)\phi(y)\,dxdy/\kappa(x) = 0$, from which follows that $\beta(x,y) \equiv 0$.

Thus it has been proved that *there are no subgroups of the conformal group which leave* (3) *invariant when* $n > 2$.

April 7, 1923.

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